Any conjecture will do: a warming-up survey

Hajo Broersma

Department of Computer Science
Durham University, UK

October 2008

Contents

Why we are here

Why we are here

- The overall motivation is to continue the workshops in Enschede, Nectiny (twice), Hannover, and Hajek in order to make progress on several intriguing conjectures.

Why we are here

- The overall motivation is to continue the workshops in Enschede, Nectiny (twice), Hannover, and Hajek in order to make progress on several intriguing conjectures.
- These highly related conjectures involve line graphs, claw-free graphs, cubic graphs, snarks, and concepts like hamiltonian cycles, dominating closed trails, dominating cycles.

Why we are here

- The overall motivation is to continue the workshops in Enschede, Nectiny (twice), Hannover, and Hajek in order to make progress on several intriguing conjectures.
- These highly related conjectures involve line graphs, claw-free graphs, cubic graphs, snarks, and concepts like hamiltonian cycles, dominating closed trails, dominating cycles.
- We have specialists in these areas here as well as interested participants that want to know more about this.

Why we are here

- The overall motivation is to continue the workshops in Enschede, Nectiny (twice), Hannover, and Hajek in order to make progress on several intriguing conjectures.
- These highly related conjectures involve line graphs, claw-free graphs, cubic graphs, snarks, and concepts like hamiltonian cycles, dominating closed trails, dominating cycles.
- We have specialists in these areas here as well as interested participants that want to know more about this.
- In order to introduce the workshop topics, I was asked to give some background, and a survey on some of the conjectures and their relationships.

The first two conjectures

The first two conjectures

The following two conjectures were tossed in the eighties.

Matthews, Sumner 1984:

Conjecture (MS)

Every 4-connected claw-free graph is hamiltonian.

Thomassen 1986:

Conjecture (T)

Every 4-connected line graph is hamiltonian.

The first two conjectures

The following two conjectures were tossed in the eighties.

Matthews, Sumner 1984:

Conjecture (MS)

Every 4-connected claw-free graph is hamiltonian.

Thomassen 1986:

Conjecture (T)

Every 4-connected line graph is hamiltonian.

Let me start by explaining the terminology to understand the above statements and their relationship.

The basics

The basics

The basics

- All graphs in this talk are finite, undirected, loopless and simple (no multiple edges).

The basics

- All graphs in this talk are finite, undirected, loopless and simple (no multiple edges).
- We denote a graph G as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.

The basics

- All graphs in this talk are finite, undirected, loopless and simple (no multiple edges).
- We denote a graph G as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.
- A graph is hamiltonian if it contains a cycle through all its vertices, i.e. a connected spanning 2-regular subgraph.

The basics

- All graphs in this talk are finite, undirected, loopless and simple (no multiple edges).
- We denote a graph G as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.
- A graph is hamiltonian if it contains a cycle through all its vertices, i.e. a connected spanning 2-regular subgraph.
- If H is a graph, then the line graph of H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two vertices in $L(H)$ are adjacent if and only if their corresponding edges in H share precisely one end vertex.

The basics

- All graphs in this talk are finite, undirected, loopless and simple (no multiple edges).
- We denote a graph G as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.
- A graph is hamiltonian if it contains a cycle through all its vertices, i.e. a connected spanning 2-regular subgraph.
- If H is a graph, then the line graph of H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two vertices in $L(H)$ are adjacent if and only if their corresponding edges in H share precisely one end vertex.
- A graph G is a line graph if it is isomorphic to $L(H)$ for some graph H.

The basics

- All graphs in this talk are finite, undirected, loopless and simple (no multiple edges).
- We denote a graph G as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.
- A graph is hamiltonian if it contains a cycle through all its vertices, i.e. a connected spanning 2-regular subgraph.
- If H is a graph, then the line graph of H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two vertices in $L(H)$ are adjacent if and only if their corresponding edges in H share precisely one end vertex.
- A graph G is a line graph if it is isomorphic to $L(H)$ for some graph H.
- Which graphs are line graphs and which are not?

A characterization of line graphs

A characterization of line graphs

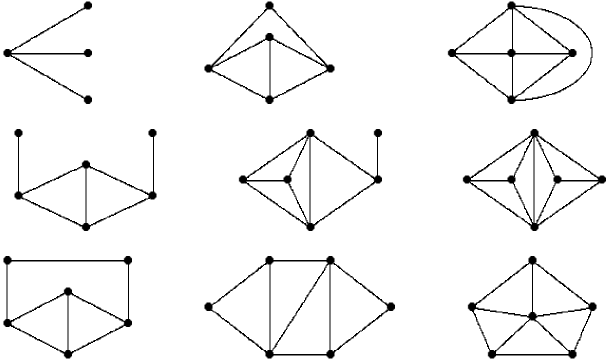
Theorem (Beineke, 1969)

A graph G is a line graph if and only if G does not contain a copy of any of the following graphs as an induced *subgraph.*

A characterization of line graphs

Theorem (Beineke, 1969)

A graph G is a line graph if and only if G does not contain a copy of any of the following graphs as an induced *subgraph.*



Forbidden induced subgraphs

Forbidden induced subgraphs

- Let G be a graph and let S be a nonempty subset of $V(G)$. Then the subgraph of G induced by S, denoted by $G[S]$, is the graph with vertex set S, and all edges of G with both end vertices in S.

Forbidden induced subgraphs

- Let G be a graph and let S be a nonempty subset of $V(G)$. Then the subgraph of G induced by S, denoted by $G[S]$, is the graph with vertex set S, and all edges of G with both end vertices in S.
- H is an induced subgraph of G if it is induced in G by some subset of $V(G)$.

Forbidden induced subgraphs

- Let G be a graph and let S be a nonempty subset of $V(G)$. Then the subgraph of G induced by S, denoted by $G[S]$, is the graph with vertex set S, and all edges of G with both end vertices in S.
- H is an induced subgraph of G if it is induced in G by some subset of $V(G)$.
- G is H-free if H is not an induced subgraph of G.

Forbidden induced subgraphs

- Let G be a graph and let S be a nonempty subset of $V(G)$. Then the subgraph of G induced by S, denoted by $G[S]$, is the graph with vertex set S, and all edges of G with both end vertices in S.
- H is an induced subgraph of G if it is induced in G by some subset of $V(G)$.
- G is H-free if H is not an induced subgraph of G.
- In particular, a graph G is *claw-free* if G does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph.

Forbidden induced subgraphs

- Let G be a graph and let S be a nonempty subset of $V(G)$. Then the subgraph of G induced by S, denoted by $G[S]$, is the graph with vertex set S, and all edges of G with both end vertices in S.
- H is an induced subgraph of G if it is induced in G by some subset of $V(G)$.
- G is H-free if H is not an induced subgraph of G.
- In particular, a graph G is *claw-free* if G does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph.
- Direct inspection or Beineke's result shows that every line graph is claw-free.

The two conjectures revisited

The two conjectures revisited

Conjecture (MS)

Every 4-connected claw-free graph is hamiltonian.

Conjecture (T)

Every 4-connected line graph is hamiltonian.

The two conjectures revisited

Conjecture (MS)

Every 4-connected claw-free graph is hamiltonian.

Conjecture (T)

Every 4-connected line graph is hamiltonian.

- Since line graphs are claw-free the first conjecture is stronger that the second one.

The two conjectures revisited

Conjecture (MS)

Every 4-connected claw-free graph is hamiltonian.

Conjecture (T)

Every 4-connected line graph is hamiltonian.

- Since line graphs are claw-free the first conjecture is stronger that the second one.
- Or are they equivalent? (A question Herbert Fleischner posed in Enschede.)

A useful tool: the closure

A useful tool: the closure

- To answer the question affirmatively, Zdeněk Ryjáček introduced a closure concept for claw-free graphs.

A useful tool: the closure

- To answer the question affirmatively, Zdeněk Ryjáček introduced a closure concept for claw-free graphs.
- It is based on adding edges without destroying the (non)hamiltonicity (similar to the Bondy-Chvátal closure).

A useful tool: the closure

- To answer the question affirmatively, Zdeněk Ryjáček introduced a closure concept for claw-free graphs.
- It is based on adding edges without destroying the (non)hamiltonicity (similar to the Bondy-Chvátal closure).
- The edges are added by looking at a vertex v and the subgraph of G induced by $N(v)$: the neighbors of v.

A useful tool: the closure

- To answer the question affirmatively, Zdeněk Ryjáček introduced a closure concept for claw-free graphs.
- It is based on adding edges without destroying the (non)hamiltonicity (similar to the Bondy-Chvátal closure).
- The edges are added by looking at a vertex v and the subgraph of G induced by $N(v)$: the neighbors of v.
- If $G[N(v)]$ is connected and not a complete graph, all edges are added to turn $G[N(v)]$ into a complete graph.

A useful tool: the closure

- To answer the question affirmatively, Zdeněk Ryjáček introduced a closure concept for claw-free graphs.
- It is based on adding edges without destroying the (non)hamiltonicity (similar to the Bondy-Chvátal closure).
- The edges are added by looking at a vertex v and the subgraph of G induced by $N(v)$: the neighbors of v.
- If $G[N(v)]$ is connected and not a complete graph, all edges are added to turn $G[N(v)]$ into a complete graph.
- This procedure is repeated in the new graph, etc., until it is impossible to add any more edges.

The two conjectures are equivalent

Theorem (Ryjáček, 1997)

Let G be a claw-free graph. Then

- *the closure* $\mathrm{cl}(G)$ *is uniquely determined,*
- $\mathrm{cl}(G)$ *is hamiltonian if and only if* G *is hamiltonian,*
- $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

The two conjectures are equivalent

Theorem (Ryjáček, 1997)

Let G be a claw-free graph. Then

- *the closure* $\mathrm{cl}(G)$ *is uniquely determined,*
- $\mathrm{cl}(G)$ *is hamiltonian if and only if* G *is hamiltonian,*
- $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Theorem (Zhan, 1991)

Every 7-connected line graph is hamiltonian.

The two conjectures are equivalent

Theorem (Ryjáček, 1997)

Let G be a claw-free graph. Then

- *the closure* $\mathrm{cl}(G)$ *is uniquely determined,*
- $\mathrm{cl}(G)$ *is hamiltonian if and only if* G *is hamiltonian,*
- $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Theorem (Zhan, 1991)

Every 7-connected line graph is hamiltonian.

Corollary

Every 7-connected claw-free graph is hamiltonian.

The two conjectures are equivalent

Theorem (Ryjáček, 1997)

Let G be a claw-free graph. Then

- *the closure* $\mathrm{cl}(G)$ *is uniquely determined,*
- $\mathrm{cl}(G)$ *is hamiltonian if and only if* G *is hamiltonian,*
- $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Theorem (Zhan, 1991)

Every 7-connected line graph is hamiltonian.

Corollary

Every 7-connected claw-free graph is hamiltonian.

There are partial results for 6-connected graphs, but the general conjectures are open for 6-connected graphs as well. The conjectures are false for 3-connected graphs.

From line graphs to their root graphs

From line graphs to their root graphs

- Whenever we consider a line graph G, we can identify a graph H such that $G = L(H)$.

From line graphs to their root graphs

- Whenever we consider a line graph G, we can identify a graph H such that $G = L(H)$.
- If G is connected this H is unique, except for $G = K_3$: then H can be K_3 or $K_{1,3}$.

From line graphs to their root graphs

- Whenever we consider a line graph G, we can identify a graph H such that $G = L(H)$.
- If G is connected this H is unique, except for $G = K_3$: then H can be K_3 or $K_{1,3}$.
- If we take $K_{1,3}$ in this exceptional case, we can talk of a unique H as the root graph of the connected line graph G isomorphic to $L(H)$.

From line graphs to their root graphs

- Whenever we consider a line graph G, we can identify a graph H such that $G = L(H)$.
- If G is connected this H is unique, except for $G = K_3$: then H can be K_3 or $K_{1,3}$.
- If we take $K_{1,3}$ in this exceptional case, we can talk of a unique H as the root graph of the connected line graph G isomorphic to $L(H)$.
- What is the counterpart in H of a hamiltonian cycle in G?

From line graphs to their root graphs

- Whenever we consider a line graph G, we can identify a graph H such that $G = L(H)$.
- If G is connected this H is unique, except for $G = K_3$: then H can be K_3 or $K_{1,3}$.
- If we take $K_{1,3}$ in this exceptional case, we can talk of a unique H as the root graph of the connected line graph G isomorphic to $L(H)$.
- What is the counterpart in H of a hamiltonian cycle in G?
- A closed trail is a connected eulerian subgraph, i.e. a connected subgraph in which all degrees are even.

From line graphs to their root graphs

- Whenever we consider a line graph G, we can identify a graph H such that $G = L(H)$.
- If G is connected this H is unique, except for $G = K_3$: then H can be K_3 or $K_{1,3}$.
- If we take $K_{1,3}$ in this exceptional case, we can talk of a unique H as the root graph of the connected line graph G isomorphic to $L(H)$.
- What is the counterpart in H of a hamiltonian cycle in G?
- A closed trail is a connected eulerian subgraph, i.e. a connected subgraph in which all degrees are even.
- A dominating closed trail (DCT) is a closed trail T such that every edge has at least one end vertex on T.

Hamiltonian cycles and dominating closed trails

Hamiltonian cycles and dominating closed trails

There is an intimate relationship between DCTs in H and hamiltonian cycles in $L(H)$.

Hamiltonian cycles and dominating closed trails

There is an intimate relationship between DCTs in H and hamiltonian cycles in $L(H)$.

Theorem (Harary and Nash-Williams, 1965)

Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.

Hamiltonian cycles and dominating closed trails

There is an intimate relationship between DCTs in H and hamiltonian cycles in $L(H)$.

Theorem (Harary and Nash-Williams, 1965)

Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.

- What is the counterpart in H of 4-connectivity in $L(H)$?

Hamiltonian cycles and dominating closed trails

There is an intimate relationship between DCTs in H and hamiltonian cycles in $L(H)$.

Theorem (Harary and Nash-Williams, 1965)

Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.

- What is the counterpart in H of 4-connectivity in $L(H)$? Note that 4-edge-connectivity is not the right answer!

Hamiltonian cycles and dominating closed trails

There is an intimate relationship between DCTs in H and hamiltonian cycles in $L(H)$.

Theorem (Harary and Nash-Williams, 1965)

Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.

- What is the counterpart in H of 4-connectivity in $L(H)$? Note that 4-edge-connectivity is not the right answer!
- A graph H is *essentially 4-edge-connected* if it contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain an edge.

Hamiltonian cycles and dominating closed trails

There is an intimate relationship between DCTs in H and hamiltonian cycles in $L(H)$.

Theorem (Harary and Nash-Williams, 1965)

Let H be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if H contains a DCT.

- What is the counterpart in H of 4-connectivity in $L(H)$? Note that 4-edge-connectivity is not the right answer!
- A graph H is *essentially 4-edge-connected* if it contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain an edge.
- $L(H)$ is 4-connected if and only if H is essentially 4-edge-connected.

Another equivalent conjecture

Another equivalent conjecture

The previous results and observations imply that the following conjecture is equivalent to the two we have seen before.

Conjecture (DCT)

Every essentially 4-edge-connected graph has a DCT.

Another equivalent conjecture

The previous results and observations imply that the following conjecture is equivalent to the two we have seen before.

Conjecture (DCT)

Every essentially 4-edge-connected graph has a DCT.

- Note that 4-edge-connected graphs contain two edge-disjoint spanning trees.

Another equivalent conjecture

The previous results and observations imply that the following conjecture is equivalent to the two we have seen before.

Conjecture (DCT)

Every essentially 4-edge-connected graph has a DCT.

- Note that 4-edge-connected graphs contain two edge-disjoint spanning trees.
- Hence 4-edge-connected graphs contain a spanning closed trail, in particular a DCT.

Another equivalent conjecture

The previous results and observations imply that the following conjecture is equivalent to the two we have seen before.

Conjecture (DCT)

Every essentially 4-edge-connected graph has a DCT.

- Note that 4-edge-connected graphs contain two edge-disjoint spanning trees.
- Hence 4-edge-connected graphs contain a spanning closed trail, in particular a DCT.
- So line graphs of 4-edge-connected graphs are hamiltonian.

Narrowing the conjectures down to cubic graphs

Narrowing the conjectures down to cubic graphs

- If H is *cubic*, i.e. 3-regular, then a DCT becomes a dominating cycle (abbreviated DC).

Narrowing the conjectures down to cubic graphs

- If H is *cubic*, i.e. 3-regular, then a DCT becomes a dominating cycle (abbreviated DC).
- A cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected.

Narrowing the conjectures down to cubic graphs

- If H is *cubic*, i.e. 3-regular, then a DCT becomes a dominating cycle (abbreviated DC).
- A cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected.
- H is *cyclically 4-edge-connected* if H contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain a cycle.

Narrowing the conjectures down to cubic graphs

- If H is *cubic*, i.e. 3-regular, then a DCT becomes a dominating cycle (abbreviated DC).
- A cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected.
- H is *cyclically 4-edge-connected* if H contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain a cycle.

Conjecture (Who can we blame for this one?)

Every cyclically 4-edge-connected cubic graph has a dominating cycle.

Narrowing the conjectures down to cubic graphs

- If H is *cubic*, i.e. 3-regular, then a DCT becomes a dominating cycle (abbreviated DC).
- A cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected.
- H is *cyclically 4-edge-connected* if H contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain a cycle.

Conjecture (Who can we blame for this one?)

Every cyclically 4-edge-connected cubic graph has a dominating cycle.

Fleischner and Jackson (1989) proved that this conjecture is equivalent to the others.

Narrowing the conjectures down to cubic graphs

- If H is *cubic*, i.e. 3-regular, then a DCT becomes a dominating cycle (abbreviated DC).
- A cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected.
- H is *cyclically 4-edge-connected* if H contains no edge-cut R such that $|R| < 4$ and at least two components of $H - R$ contain a cycle.

Conjecture (Who can we blame for this one?)

Every cyclically 4-edge-connected cubic graph has a dominating cycle.

Fleischner and Jackson (1989) proved that this conjecture is equivalent to the others.

Main ingredient: Let H be an essentially 4-edge-connected graph of minimum degree $\delta(G) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Then some inflation of H at v is essentially 4-edge-connected.

Let's blame Herbert for the next one

Let's blame Herbert for the next one

Restricting ourselves to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Herbert Fleischner.

Let's blame Herbert for the next one

Restricting ourselves to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Herbert Fleischner.

Conjecture

Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a dominating cycle.

Let's blame Herbert for the next one

Restricting ourselves to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Herbert Fleischner.

Conjecture

Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a dominating cycle.

Kochol (2000) proved that this conjecture is also equivalent to the others.

Let's blame Herbert for the next one

Restricting ourselves to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Herbert Fleischner.

Conjecture

Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a dominating cycle.

Kochol (2000) proved that this conjecture is also equivalent to the others.

A *snark* is a cyclically 4-edge-connected cubic graph of girth at least 5 that is not 3-edge-colorable.

Let's blame Herbert for the next one

Restricting ourselves to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Herbert Fleischner.

Conjecture

Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a dominating cycle.

Kochol (2000) proved that this conjecture is also equivalent to the others.

A *snark* is a cyclically 4-edge-connected cubic graph of girth at least 5 that is not 3-edge-colorable.

Conjecture

Every snark has a dominating cycle.

Let's blame Herbert for the next one

Restricting ourselves to cyclically 4-edge-connected cubic graphs that are not 3-edge-colorable, we obtain the following conjecture posed by Herbert Fleischner.

Conjecture

Every cyclically 4-edge-connected cubic graph that is not 3-edge-colorable has a dominating cycle.

Kochol (2000) proved that this conjecture is also equivalent to the others.

A *snark* is a cyclically 4-edge-connected cubic graph of girth at least 5 that is not 3-edge-colorable.

Conjecture

Every snark has a dominating cycle.

In one of the later talks, it is shown that the above conjecture is also equivalent to the others.