Three problems on dominating cycles

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Partly based on joint work with M. Kriesell, R. Škrekovski and P. Vrána

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Trails through vertices of large degree

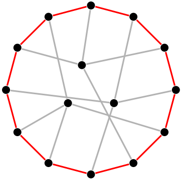
Conjecture (Jackson, 1995)

Every essentially 6- (or 5- or 4-)edge-connected graph admits a closed trail through all vertices of degree at least 4.

Trails through vertices of large degree (cont'd)

Problem

Let G be an essentially 4-edge-connected graph with an independent set X such that $G - X$ is a cycle, $d(v) \geq 4$ if $v \in X$, and $d(v) = 3$ otherwise. Does G admit a closed trail spanning X?



Need to assume some edge-connectivity:

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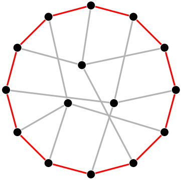

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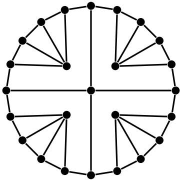

Eulerian subgraphs and cuts

- G is **eulerian** $=$ all degrees even (need not be connected)

Observation

An eulerian subgraph T of an essentially 4-edge-connected cubic graph is a dominating cycle iff for every minimal edge-cut C of size $|C| \geq 4$,

$$E(T) \cap C \neq \emptyset.$$

Conjecture (TK, Škrekovski, 2008)

Every graph admits an eulerian subgraph T intersecting all minimal edge-cuts C with $|C| \in \{4, 5, 6, \ldots\}$.

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Every graph admits an eulerian subgraph intersecting all minimal edge-cuts C with $|C| \in \{3, 4\}$.

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The Four Color Theorem is equivalent to the statement that every planar graph admits an eulerian subgraph intersecting all minimal edge-cuts C with $|C| \in \{3, 5, 7, \ldots\}$.

Question

Does every graph admit an eulerian subgraph intersecting all minimal edge-cuts C with $|C| \geq 7$? (Or 100?)

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Disjoint spanning trees

Theorem

If G contains 2 edge-disjoint spanning trees, then G contains a spanning closed trail.

Proof idea:

- let X be the set of vertices of odd degree in tree T_1
- construct $H \subset T_2$ by including an edge e in H iff each component of $H - e$ has odd intersection with X
- check that $T_1 \cup H$ is connected and has even degrees

By Tutte and Nash-Williams' characterization of graphs with two disjoint spanning trees:

- every 4-edge-connected graph contains a SCT,
- every essentially 7-edge-connected graph contains a SCT.

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(3,5)-regular bipartite graphs

Problem (Kriesell?)

Let G be an essentially 6-edge-connected bipartite graph with color classes V_1 and V_2. Assume that the degree of each vertex is

$$d(v) = \begin{cases} 3 & \text{if } v \in V_1, \\ 5 & \text{if } v \in V_2. \end{cases}$$

Does G admit a spanning closed trail covering V_2?

- G need not have two disjoint spanning trees

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Spanning hypertrees

- idea: consider a hypergraph H on V_2 with one edge of size 3 for each vertex in V_1
- what structure in H would yield a 'good' pair of trees?

Definition

A **spanning hypertree** in H is a subhypergraph W such that we may choose two vertices x_e, y_e from each hyperedge $e \in W$ such that the edges $x_e y_e$ ($e \in W$) form a spanning tree on V_2.

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- H is k-**edge-connected** if the removal of $\leq k-1$ hyperedges does not disconnect it
- our H is 5-edge-connected and has no nontrivial 6-edge-cuts

Theorem (Frank, Király and Kriesell)

A (kq)-edge-connected hypergraph H of rank q contains k disjoint spanning hypertrees.

- this is not enough: to contain two disjoint spanning hypertrees, H would have to be 6-edge-connected...

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An improvement

Observation

It suffices to find disjoint trees T and T' spanning V_2 such that no vertex of V_1 has degree 3 in T.

- for T, we do need a spanning hypertree of H
- but for T', 'any spanning connected subhypergraph will do'

Problem

Let H be a 3-uniform 5-edge-connected hypergraph with no nontrivial 6-edge-cut. Does H contain a spanning hypertree T_1 and a spanning connected subhypergraph T_2 such that T_1 and T_2 are edge-disjoint?

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