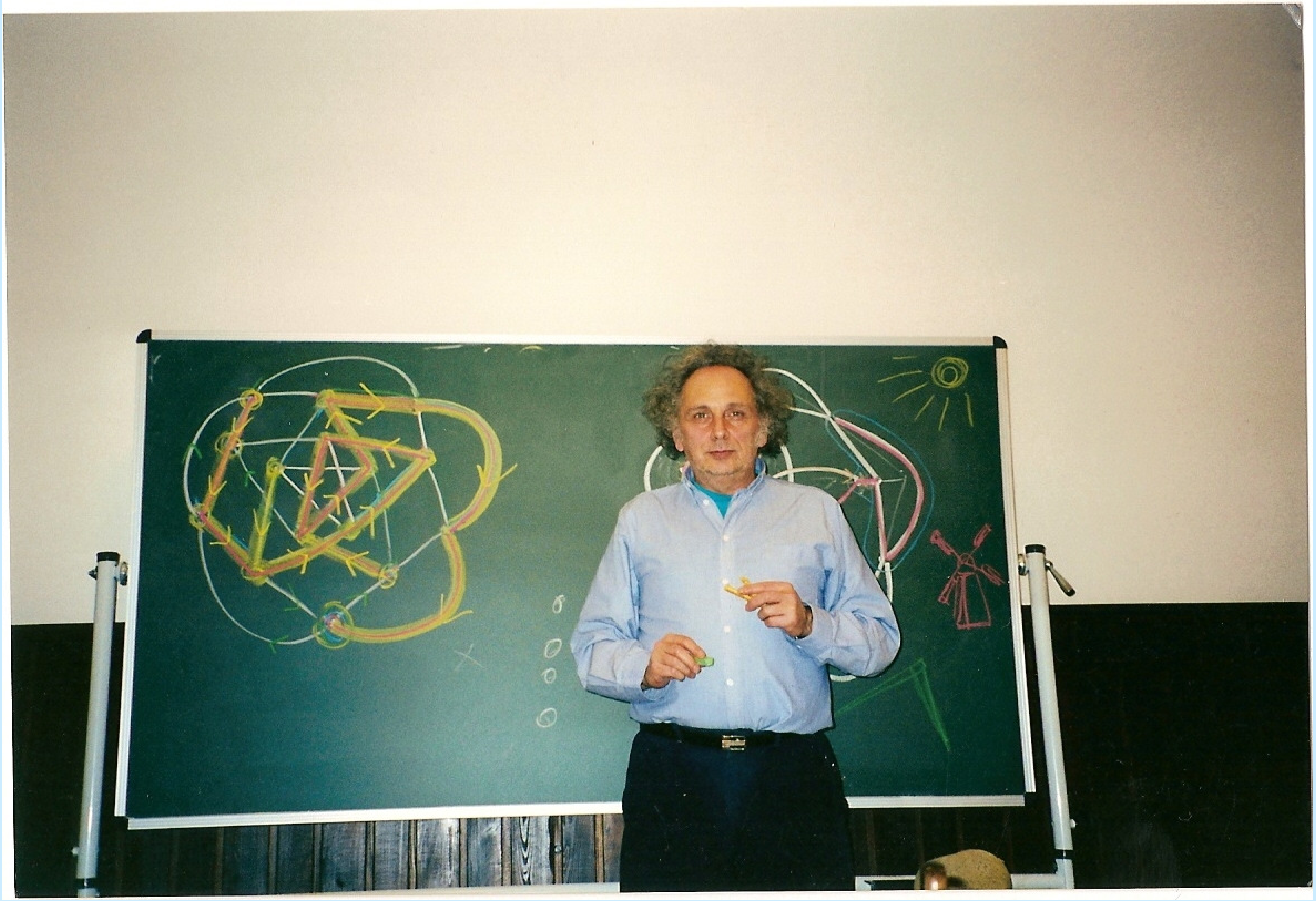
Cyclability in graphs

Hao LI

CNRS - Université de Paris-sud XI



Definition: ***cyclable* , *cyclability***

- Let S be a subset of vertices. If the graph has a cycle C containing all vertices of S, we say that S is *cyclable*.

- The ***cyclability*** of a graph is the maximum number cyc(G) such that every subset of cyc(G) vertices is cyclable.
- V. Chvátal, New directions in hamiltonian graph theory, New Directions in the Theory of Graphs, Ed.: F. Harary, Academic Press, New York, 1973, 65-95.

Cyclable

Theorem: (Bollobás and Brightwell) Let G be a graph of order n and let S $\subseteq$ V(G), s=|S|, such that the degree of every vertex in S is at least d. then there is a cycle throught at least $\lfloor s/(n/d -1) \rfloor$ vertices of S.

Cyclability, cubic graphs

Theorem:

cyc(G) ≥9 for every 3-connected cubic graph G.

This bound is sharp (The Petersen graph).

D.A. Holton, B.D. McKay, M.D. Plummer and C. Thomassen, A nine-point theorem for 3-connected graphs, Combinatorica, 2, 1982, 53-62.

Theorem:

If G is 3-connected and planar, cyc(G) ≥ 23.

This bound is sharp.

R.E.L. Aldred, S. Bau, D.A. Holton and B. McKay, Cycles through 23 vertices in 3-connected cubic planar graphs, Graphs Combin., 15, 1999, 373-376.

Cyclability, claw-free graphs

Theorem:

If G is 3-connected and claw-free, then cyc(G)≥ 6.

L.R. Markus, Degree, neighbourhood and claw conditions versus traversability in graphs, Ph.D. Thesis, Department of Mathematics, Vanderbilt University, 1992.

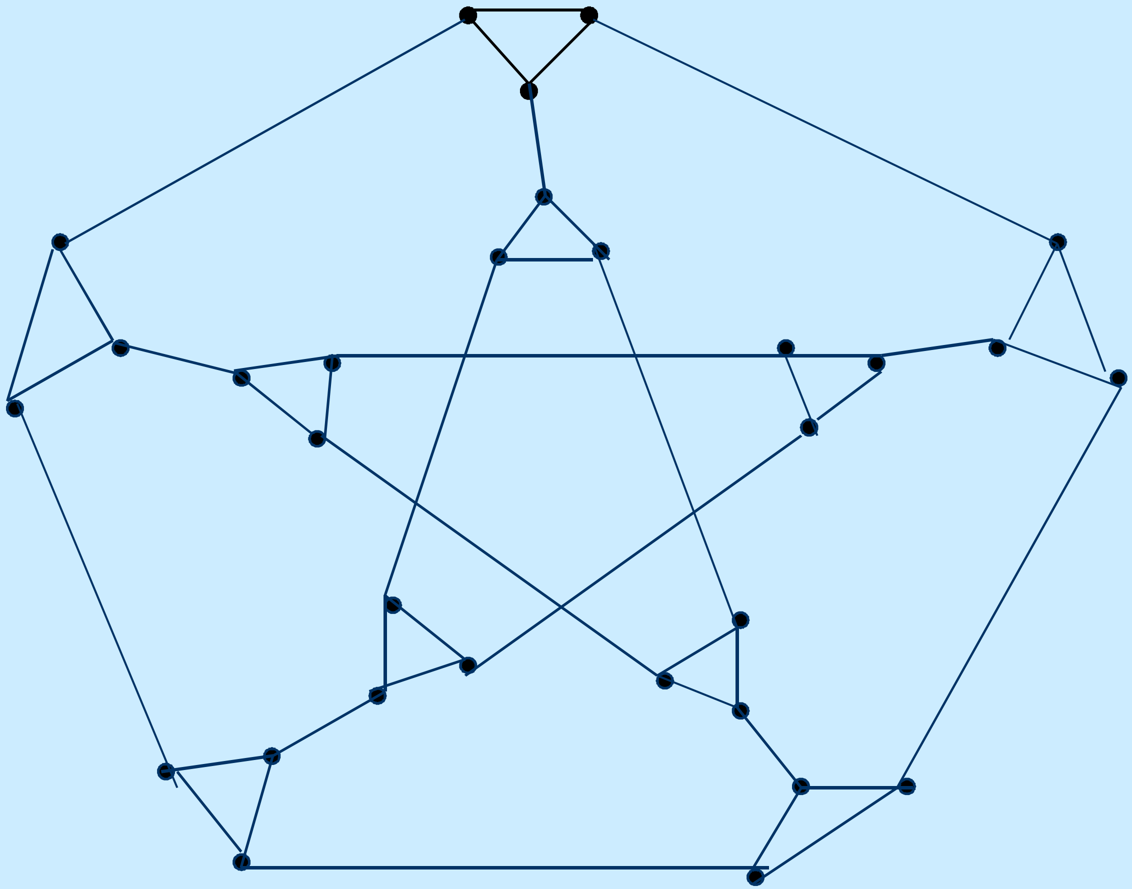
Theorem: Let G be a 3-connected claw-free graph and let U= $\{u_1,\ldots,u_k\}$, $k\leq 9$, be an arbitrary set of at most nine vertices in G. Then G contains a cycle C which contains U.

A NINE VERTEX THEOREM FOR 3-CONNECTED CLAW-FREE GRAPHS,

Ervin Gyori and Michael D. Plummer

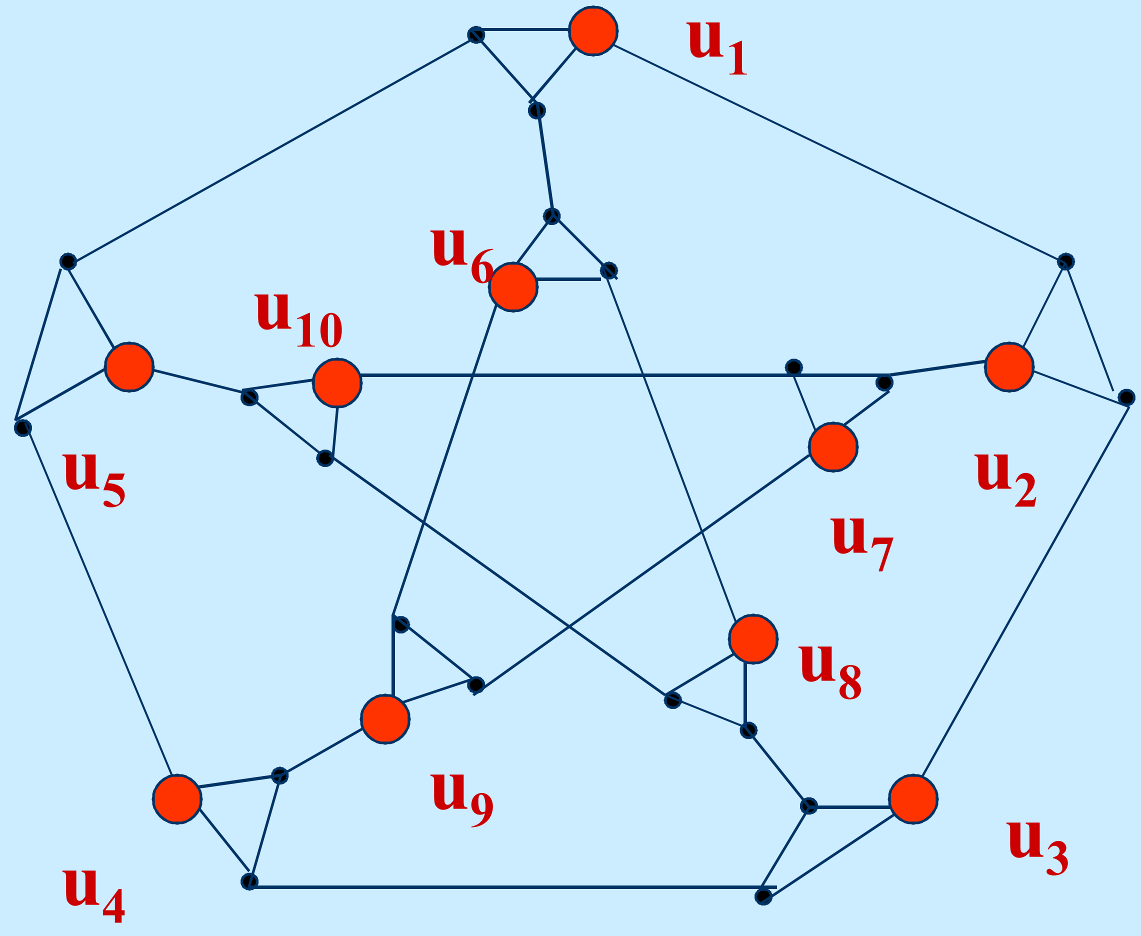
Independently by Jackson and Favaron ??

Example: Sharpness.



3-connected claw-free,

Example: Sharpness.



3-connected claw-free, cyc(G) ≤ 9

New result.

Theorem (Flandrin, Gyori, Li, Shu,):

Let G be a $K_{1,4}$-free graph and S a k-connected subset of vertices in G. Then if $|S| \leq 2k$, there exists a cycle containing S.

Corollary.

If G is $K_{1,4}$-free k-connected, then the cyclability cyc(G) is at least $2k$.

New result.

Theorem (Zhang, Li):

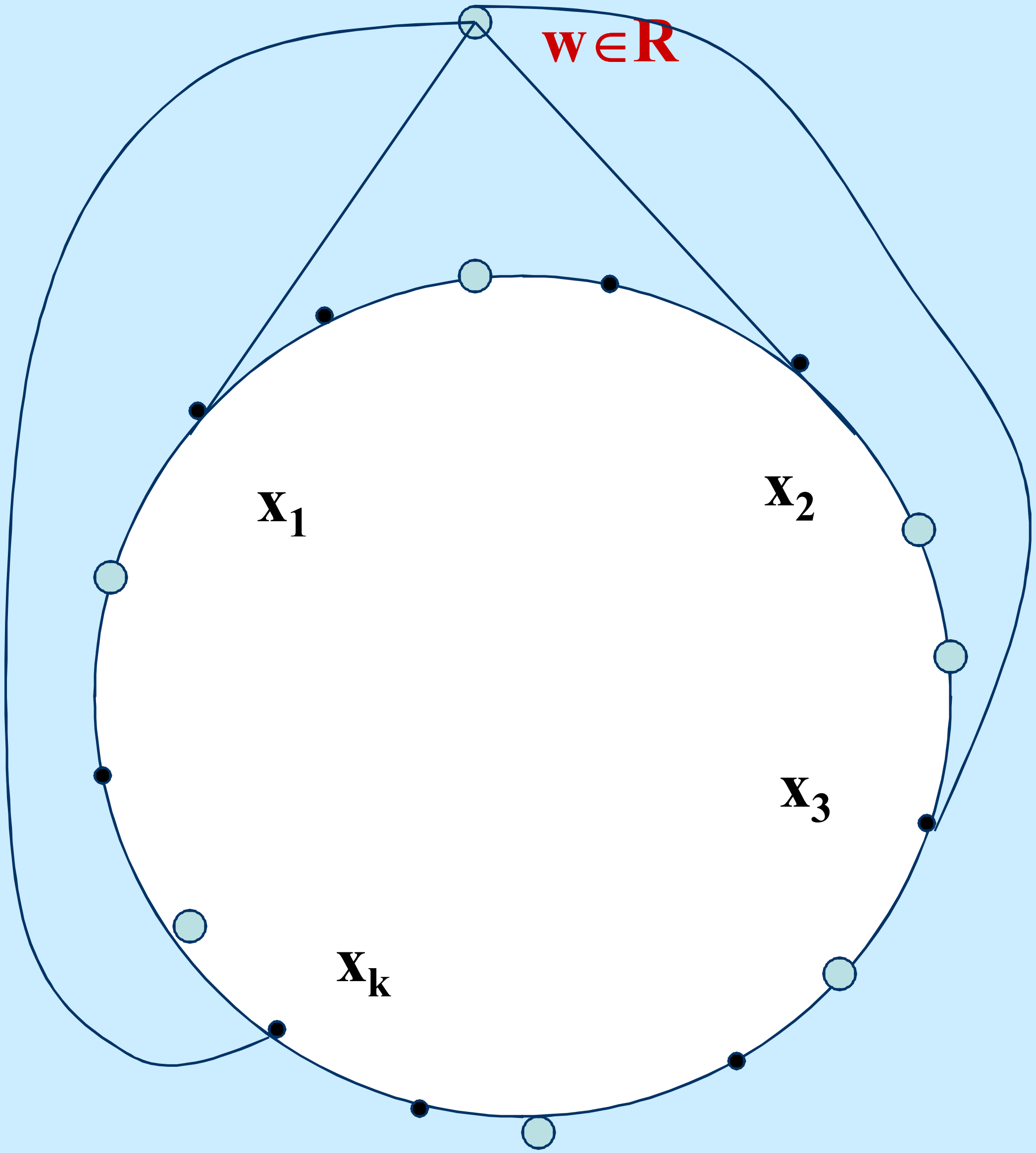
Let *G* be a 3-edge-connected graph and *S* a weak-k-edge connected vertex subset of vertices in *G with* $1 \le |S| \le 2k$. Then G admits an eulerian subgraph containing all vertices of S

A vertex set S $\subseteq$ V(G) is weak-k-edge-connected if for every subset C of S and x $\in$ S-C, there are min{k,|C|} edge-disjoint (x,C)-paths in G.

The condition " 3-edge-connected" is necessary:

G=$K_{2,2m+1}$and S is the (2m+1)- part.

Proof: ideas

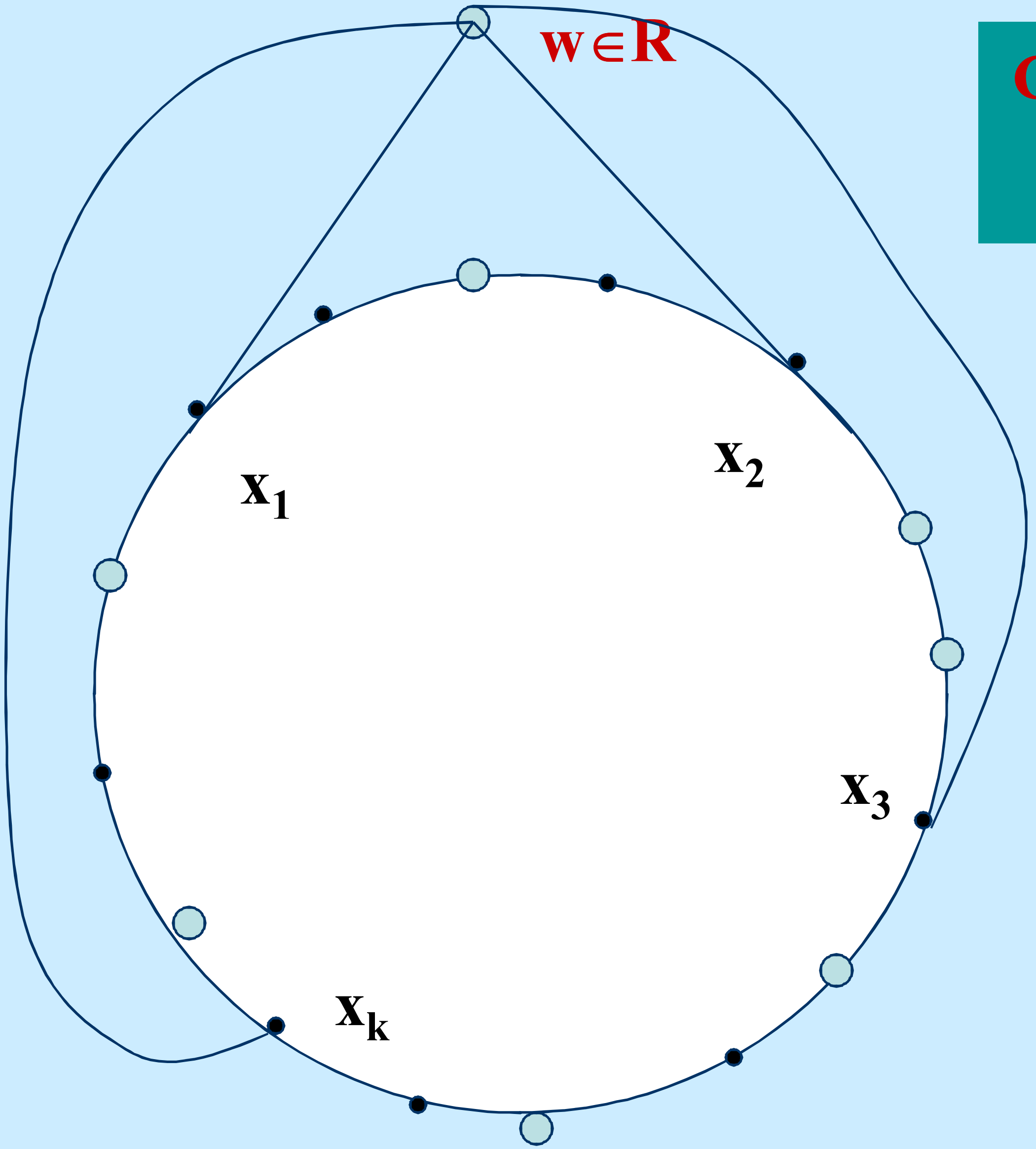


C: $|S \cap C| = \max$

G is k-connected

- $|S| \geq 3$, $|S \cap C| \geq 2$
- $|S \cap C(x_i, x_{i+1})| \geq 1$,
- $|S \cap C| \geq k$

Proof: ideas



C: (1) $|S\cap C|$=max $\geq$ **k**

(2) subject to (1), $|C|$=max

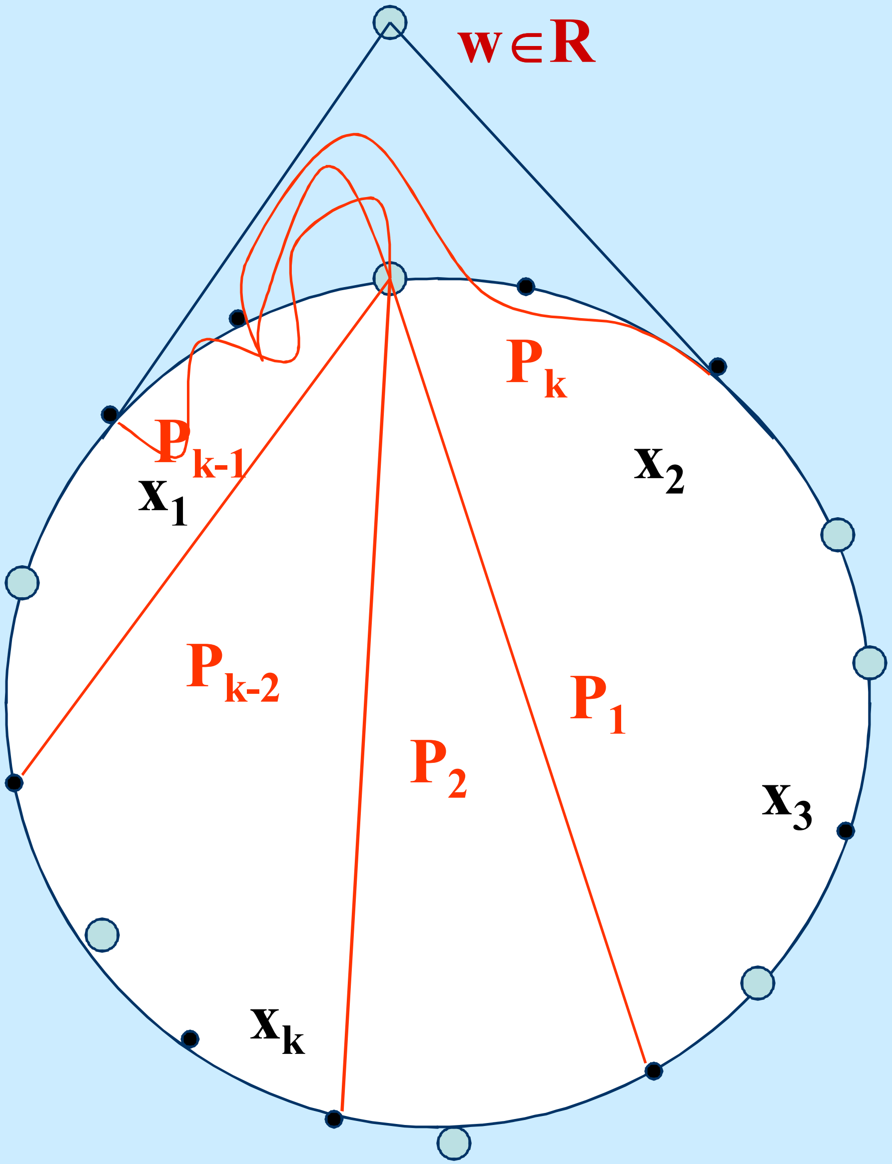
In the proof. A lemma

Lemma: Let G be graph, $M \subseteq V(G)$ with $|M| \geq k$, $v \notin M$ such that $M \cup \{v\}$ is k-connected. Suppose that there are two internal disjoint paths $Q_j[v,x_j]$, $1 \leq j \leq 2$ from v to x_1 and x_2 such that all inner vertices of these paths are in V(G)-M. Then there are k paths $P_i[v,u_i]$, $1 \leq i \leq k$ from v to $\{ u_i : 1 \leq i \leq k\} \subseteq M \cup \{x_1, x_2\}$, with $u_{k-1} = x_1$ and $u_k = x_2$ such that

1) all inner vertices of these paths are in V(G)-M,

2) $P_1[v,u_1], P_2[v,u_2], P_3[v,u_3], \ldots\ldots, P_{k-2}[v,u_{k-2}], P_{k-1}[v,u_{k-1}]$ and $P_1[v,u_1], P_2[v,u_2], P_3[v,u_3], \ldots\ldots, P_{k-2}[v,u_{k-2}], P_k[v,u_k]$ are pairwise internal disjoint respectively .

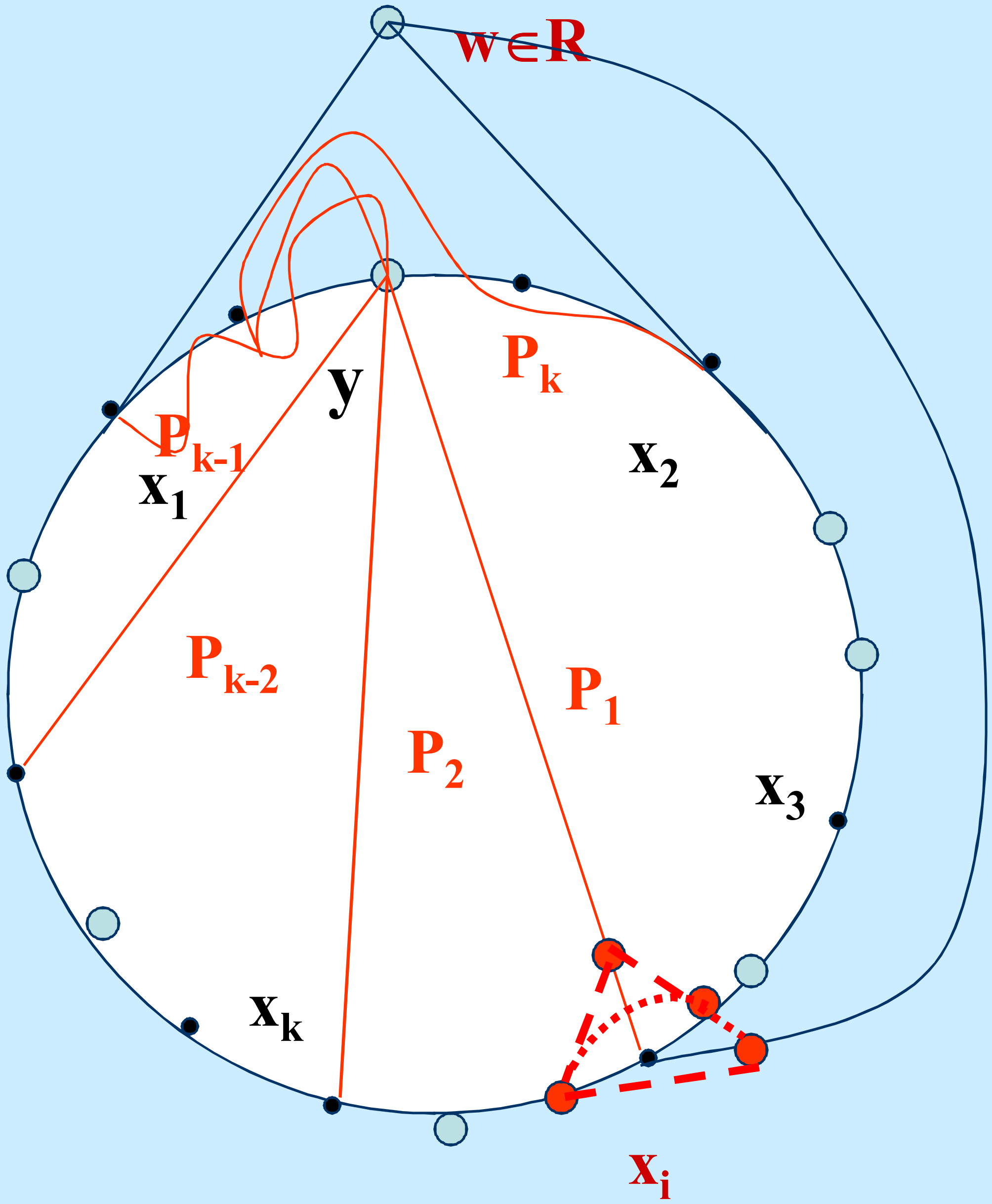
Proof: ideas



By using the lemma, get

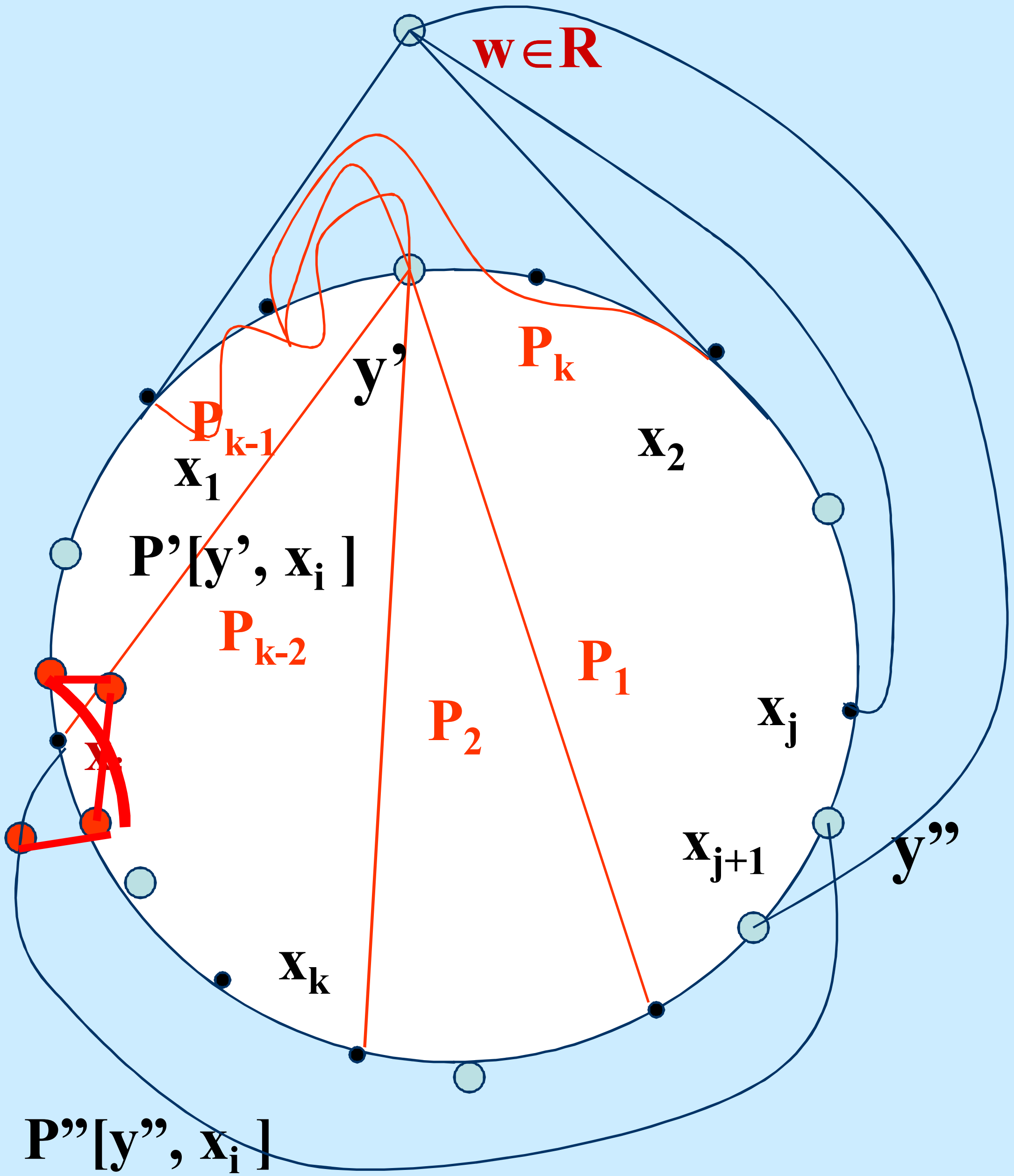
$P_1\, P_2 \ldots P_{k-2}\, P_{k-1}\, P_k$

Proof: ideas



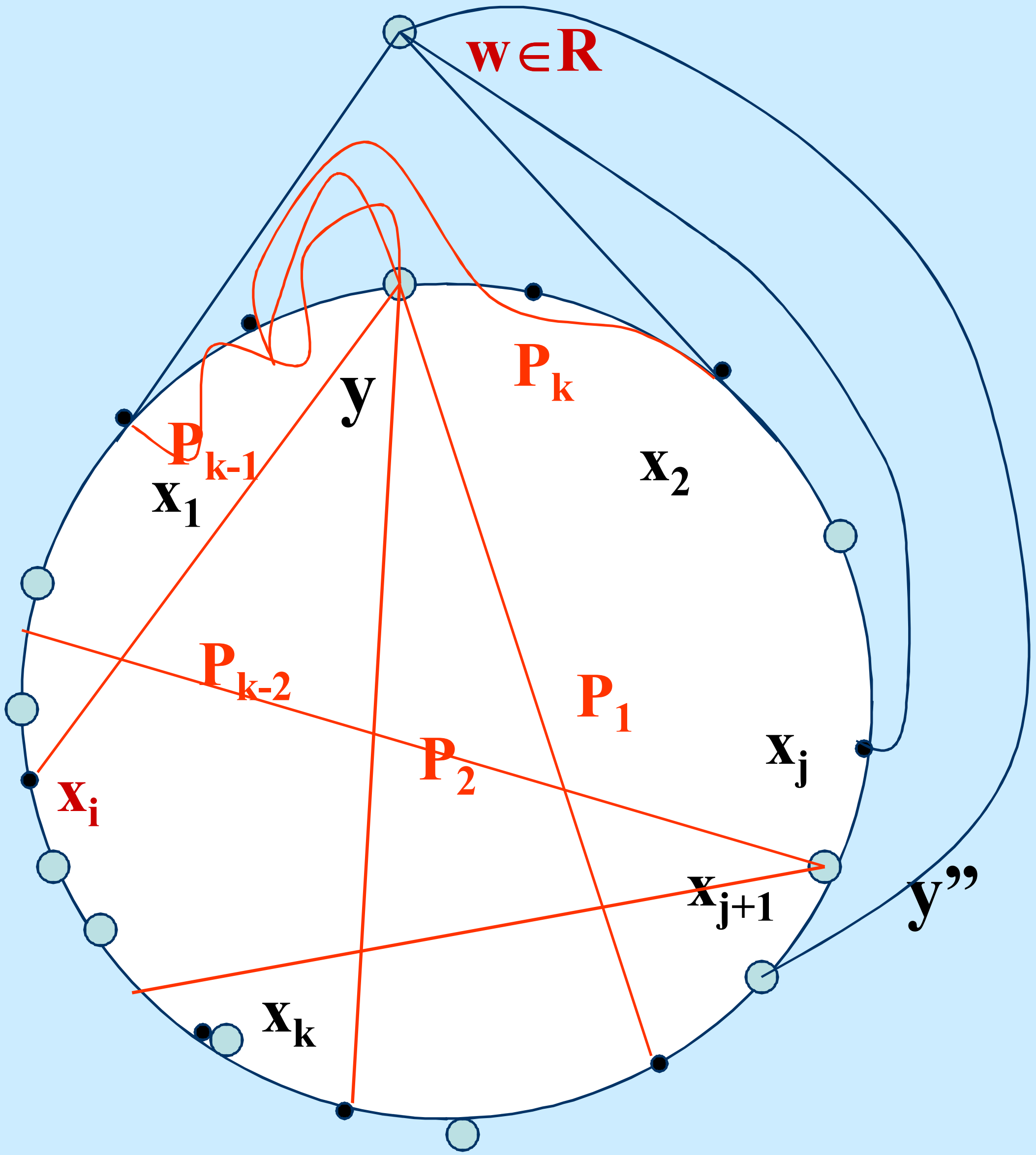
Let $y = S \cap C(x_1,x_2)$. If x_i is the end of a path P_j , $3 \le i \le k$, $1 \le j \le k-2$, it follows that the 4 red vertices are independent and together with x_i they make a $K_{1,4}$!

Proof: ideas



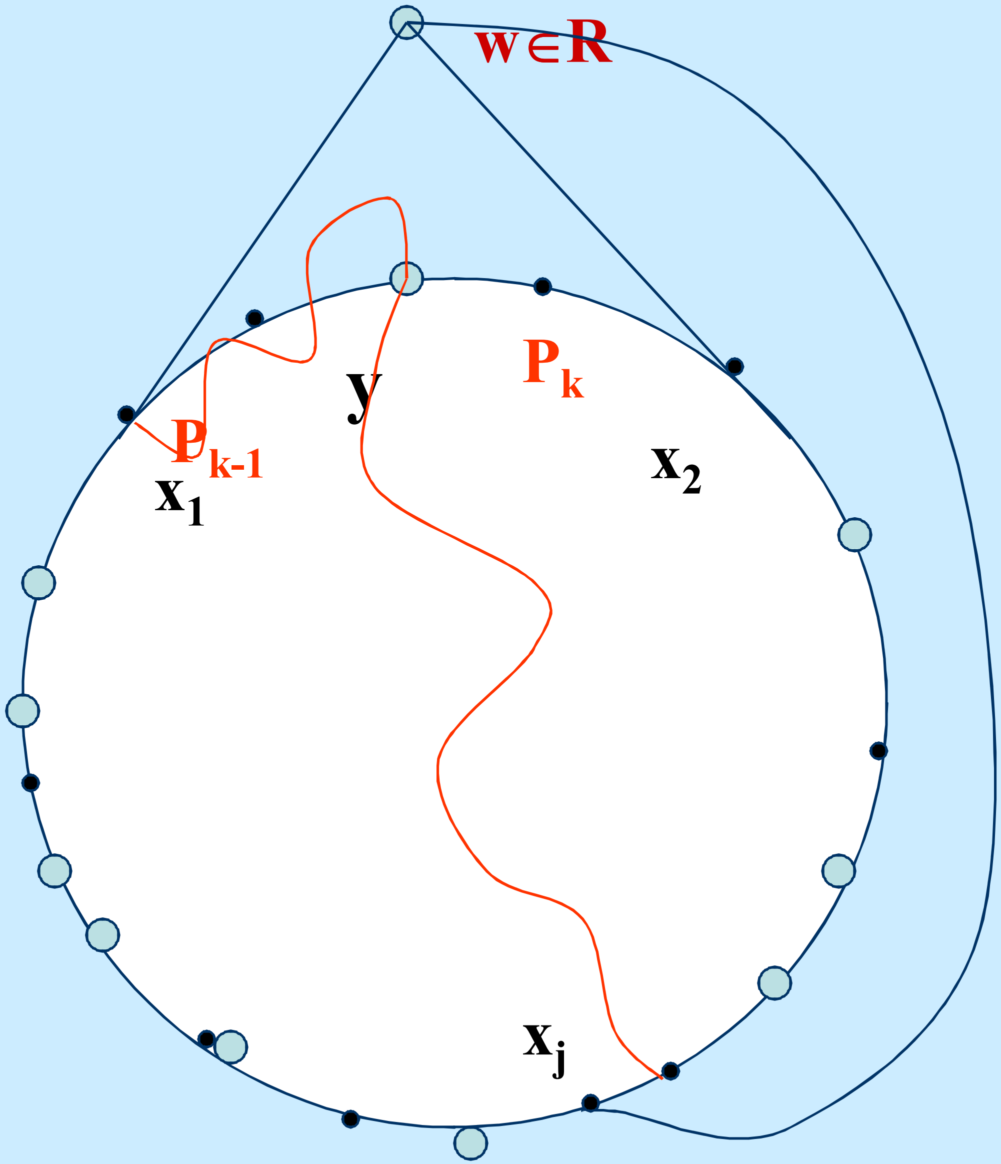
Let y' =S∩C(x_1,x_2) and y"=S ∩C(x_j,x_{j+1}). If x_i is the end of both paths P'[y', x_i] and P"[y", x_i], it follows again that the 4 red vertices are independent and together with x_i they make a $K_{,1,4}$!

Proof: ideas



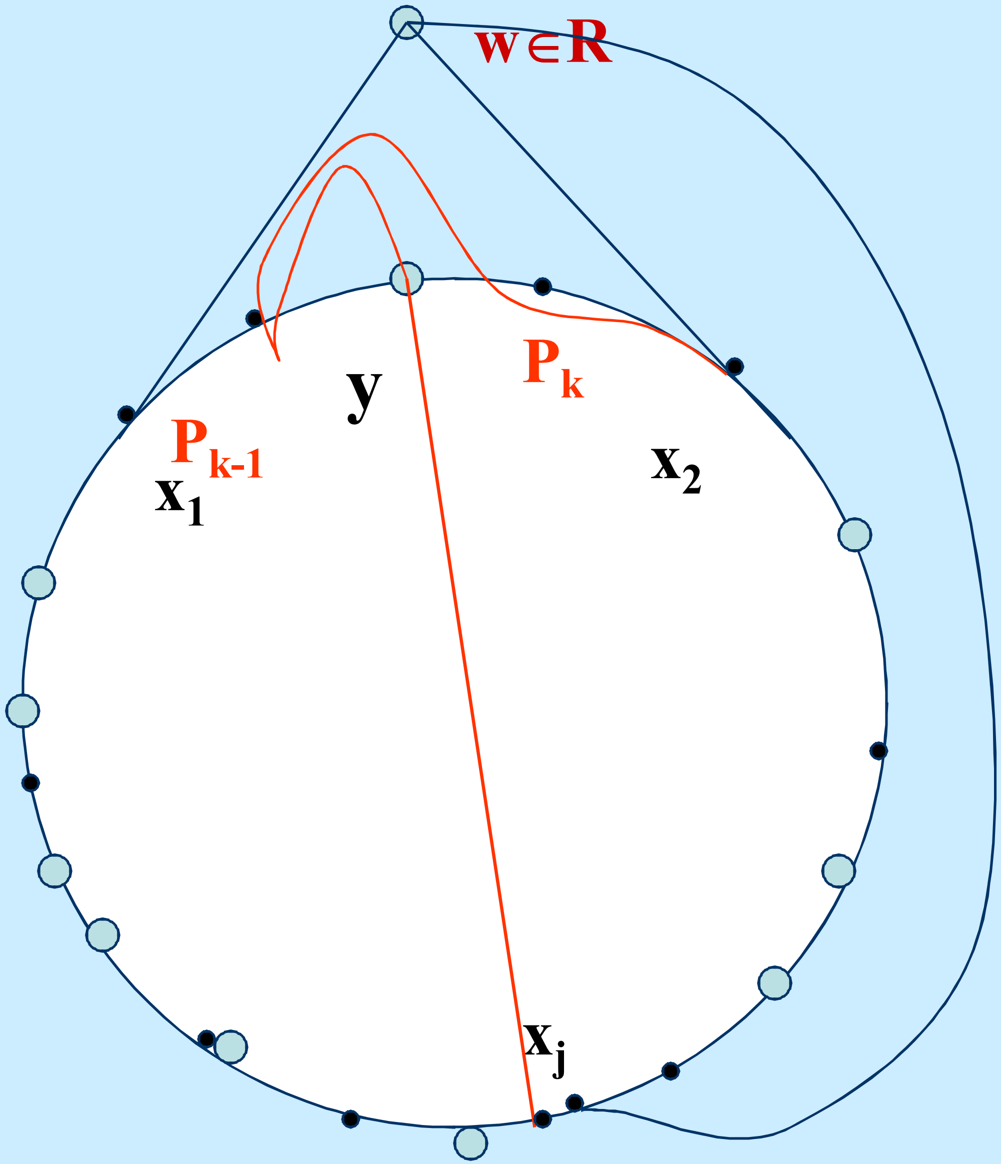
There is at least one S vertex in every segment cut by the vertices of $\{x_1,x_2,\ldots,x_k\}\cup(\cup_y N^*(y))$ since otherwise there is a cycle contains at least one more S vertex than C.

Proof: ideas



There is at least one S vertex in every segment cut by a vertex of N*(y') and a vertex in $\{x_1, x_2, \ldots, x_k\}$ since otherwise there is a cycle contains at least one more S vertex than C.

Proof: ideas



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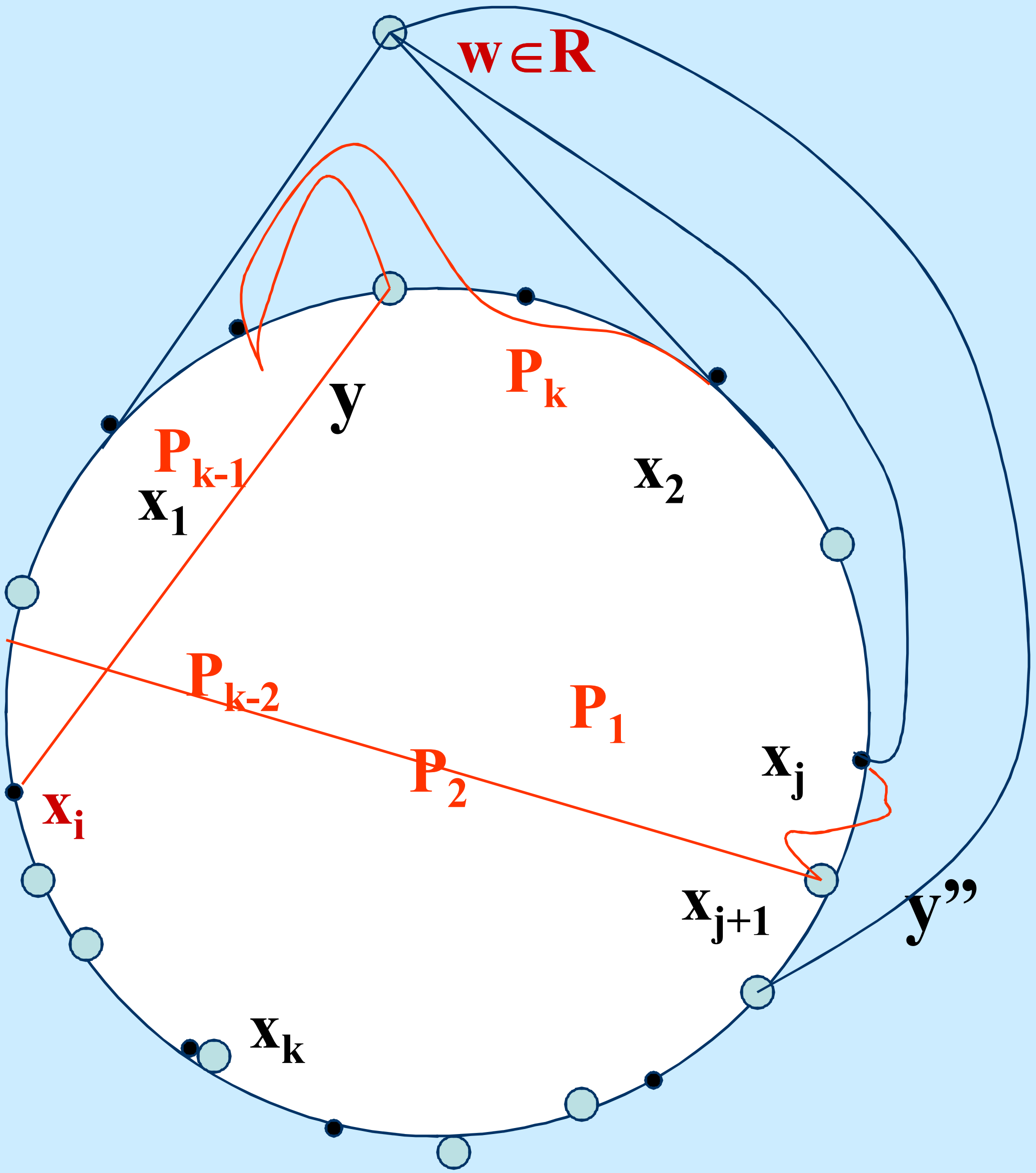
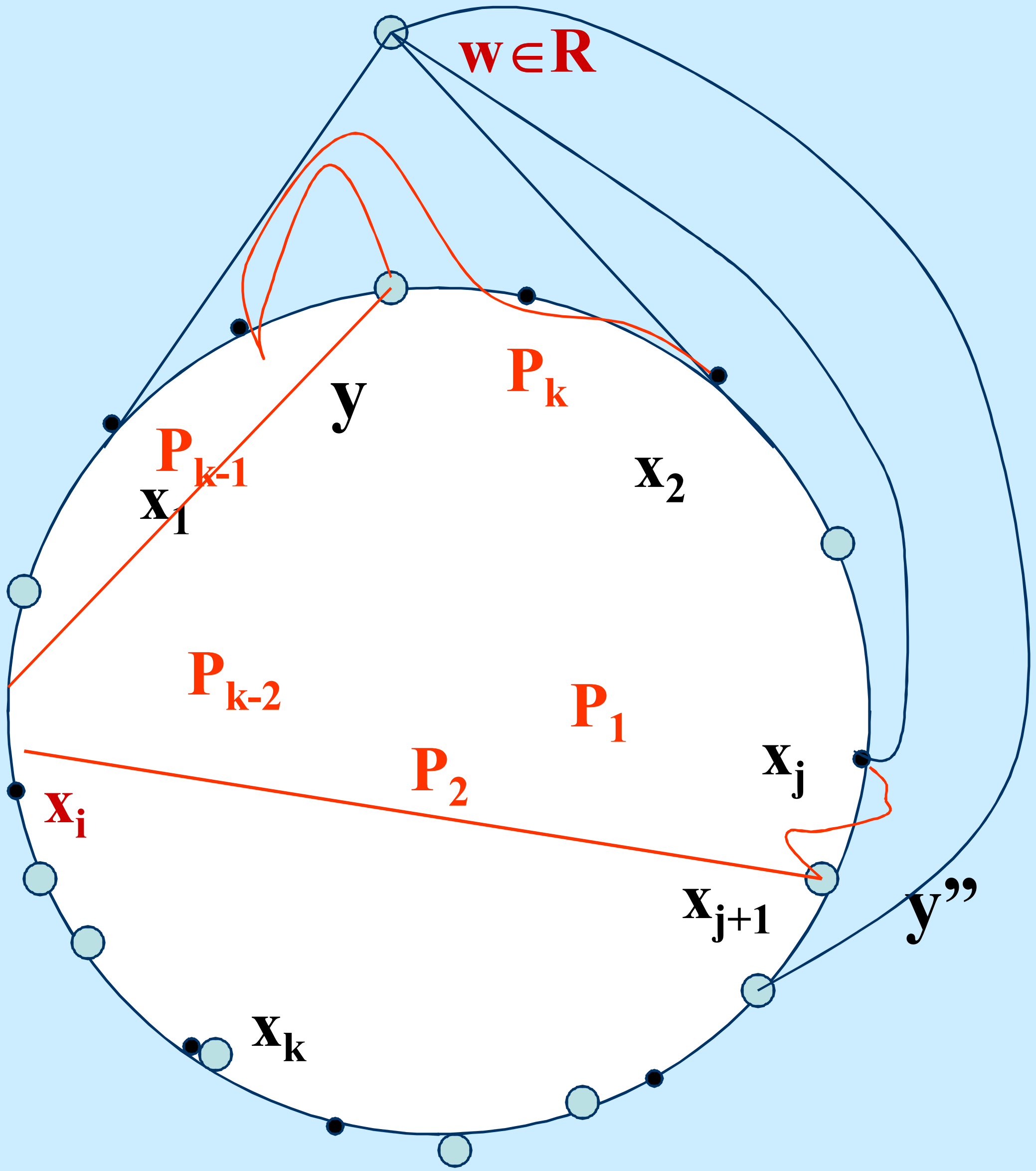
Proof: ideas



There is at least one S vertex in every segment cut by a vertex in N*(y’) and a vertex in N*(y”)

since otherwise there is a cycle contains at least one more S vertex than C.

Proof: ideas



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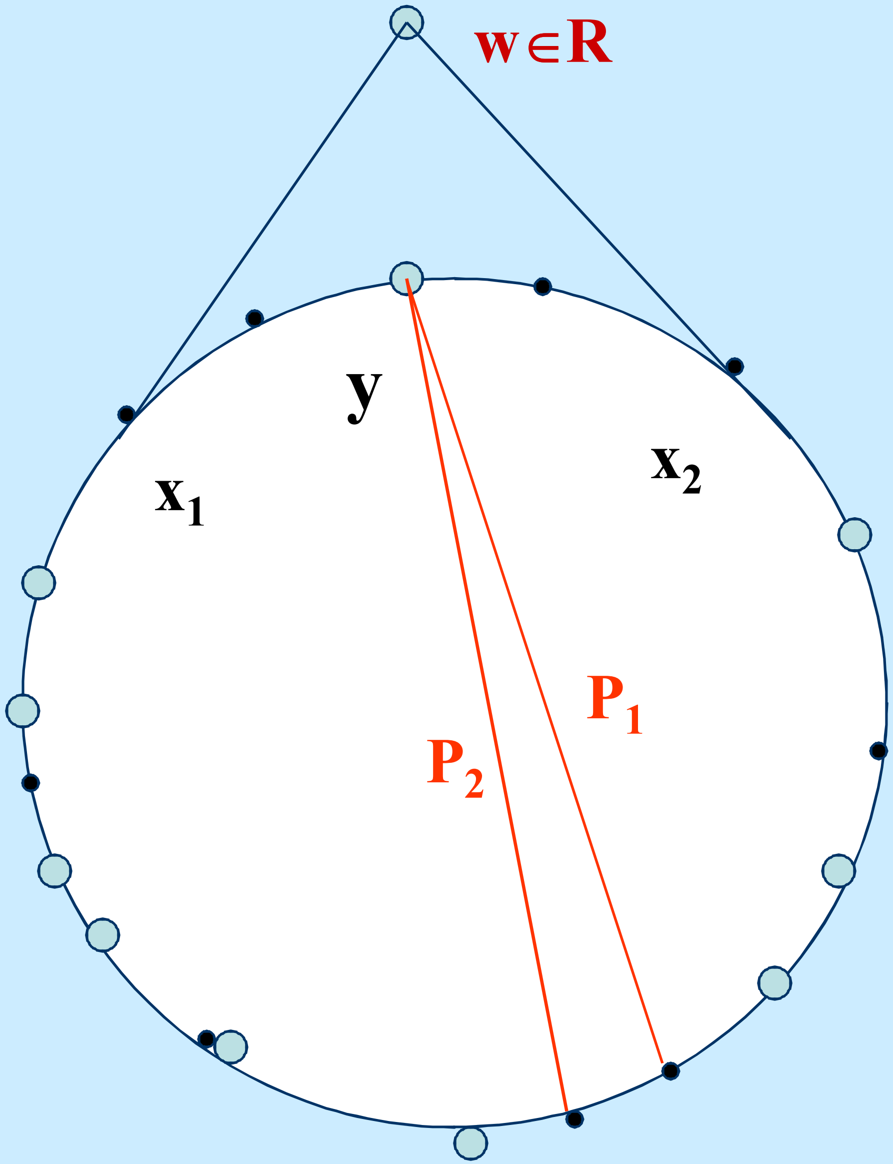
Proof: ideas



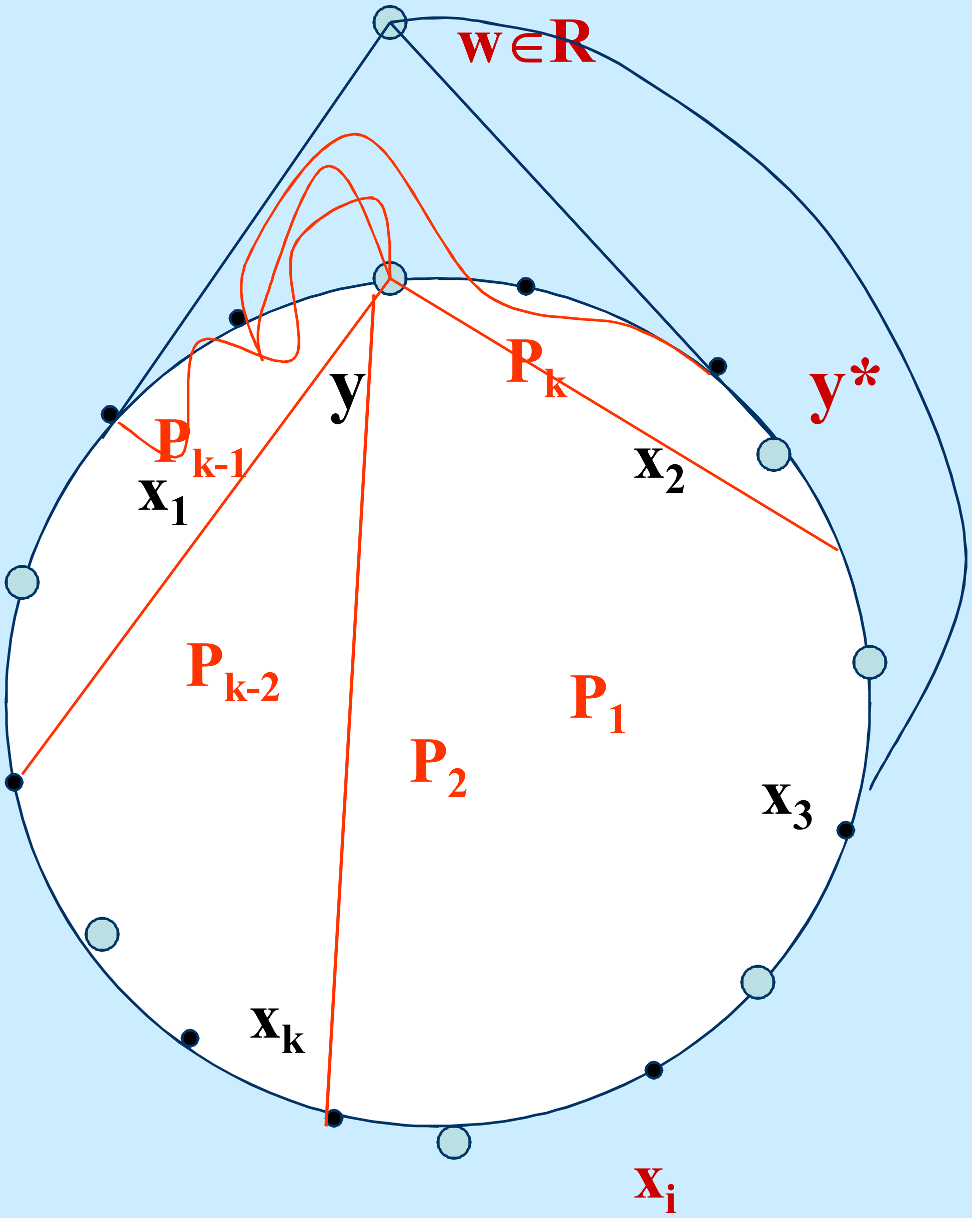
There is at least one S vertex in every segment cut by two vertices of N*(y) since otherwise there is a cycle contains at least one more S vertex than C.

So we have

$$|S \cap C| \geq k+|Y|(k-2).$$

And if $|Y|>1$, we are done

Proof: ideas

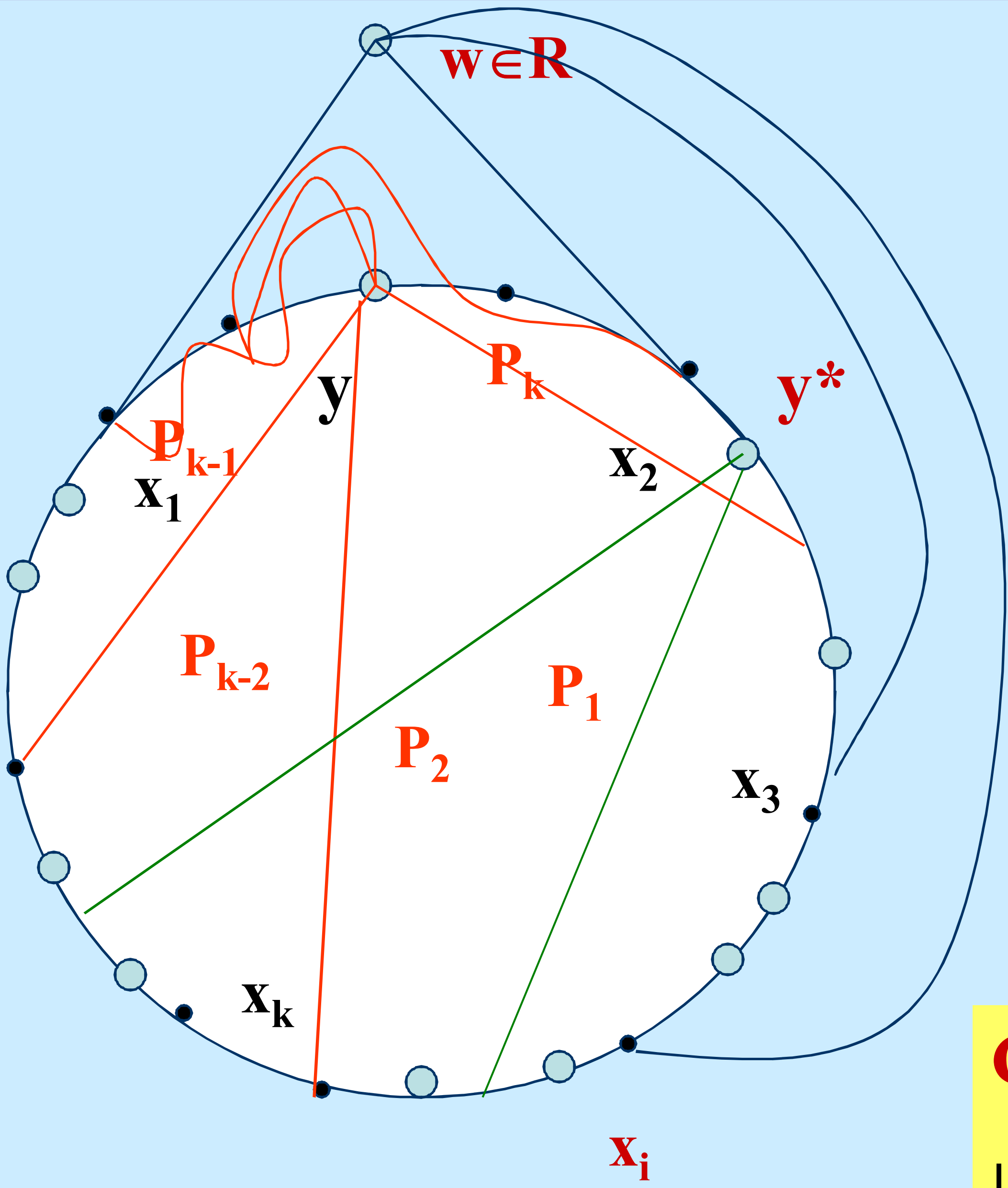


Suppose $|Y|=1$ and $y = S \cap C(x_1,x_2)$.

- **Every other segment has exactly two S vertices.**
- **WLOG,**

  $\exists\, y^* \in C(x_2,x_3)$.

Proof: ideas



- By using the lemma, get paths

  $Q_1\ Q_2\ \ldots Q_{k-2} Q_{k-1}\ Q_k$

- The end vertices of the Q_i 's are distinct to $\{x_1,x_2,\ldots, x_k\} \cup N^*(y)$
- There is at least one S vertex in every segment cut by the vertices of $\{x_1,x_2,\ldots, x_k\} \cup N^*(y) \cup N^*(y^*)$

Conclusion:

$|S \cap C| \geq k+2(k-2) \geq 2k.$

结束语=the end

谢谢=THANKS!

Thanks