Hypohamiltonian cubic graphs

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Hamiltonian graphs

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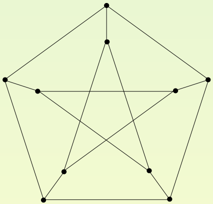
- "nearly" hamiltonian graphs:
  - maximally non-hamiltonian graphs
    $G + e$ is hamiltonian for every e
  - hypohamiltonian (HH)
    $G - v$ is hamiltonian for every v

Hypohamiltonian graphs

- Petersen graph is the smallest among HH



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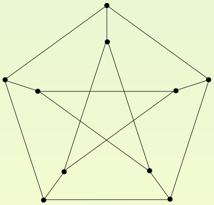

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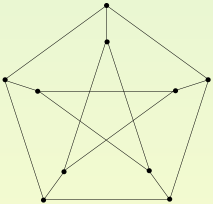

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  $\Rightarrow$ minimum valency is 3
- no HH graph can be bipartite

Hypohamiltonian graphs

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- Thomassen's construction (1974)
- Chvátal (1972): *Does there $\exists$ a planar hypohamiltonian graph?*
- infinitely many planar hypohamiltonian graphs (order ≥ 105)
  [Thomassen 1976]

Cubic hypohamiltonian graphs

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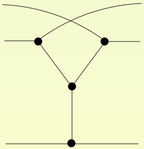
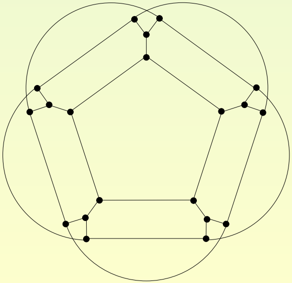
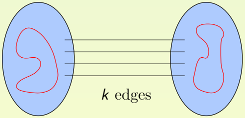
Example: Isaacs flower snarks [Fiorini 1983]

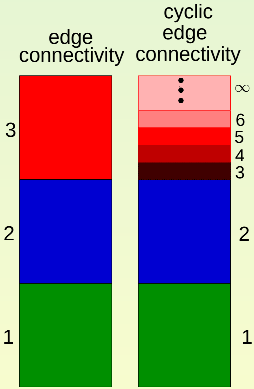




Cyclic connectivity and girth

Cyclic connectivity (cc) = smallest number of edges to be removed in order to obtain at least two components containing circuits





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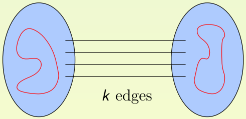

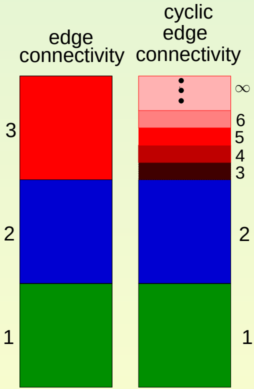

- *cc* $\leq$ girth

Cyclic connectivity and girth of cubic HH graphs

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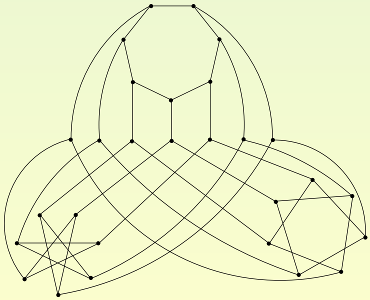

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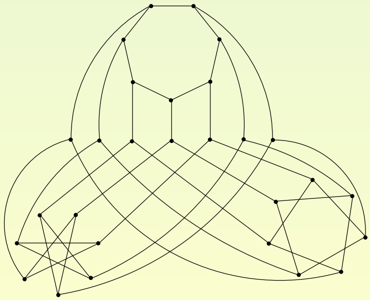

Coxeter graph

- cubic
- 3-edge-colourable
- HH
- $cc = 7$, $g = 7$

Known HH cubic graphs according to *cc* and *g*

cc \ g	4	5	6	7	
4					
5	X				
6	X	X			
7	X	X	X		

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cyclic connectivity – conjectures

Conjecture (Jaeger, Swart'80)

There are no snarks with cyclic connectivity greater than 6.

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Conjecture (Thomassen)

There exists a constant k (possibly $k = 8$) such that every cyclically k-edge-connected cubic graph is hamiltonian.

Known HH cubic graphs according to *cc* and *g*

g / cc	4	5	6	7
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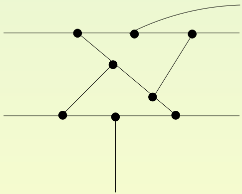

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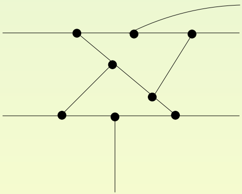

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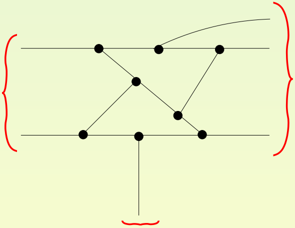

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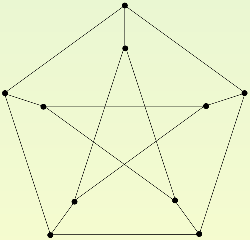

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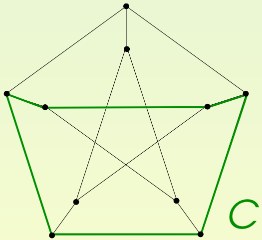

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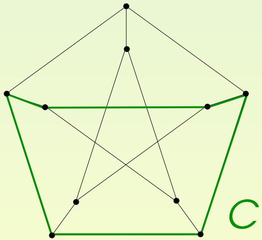

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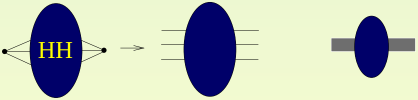

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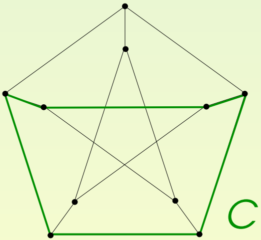

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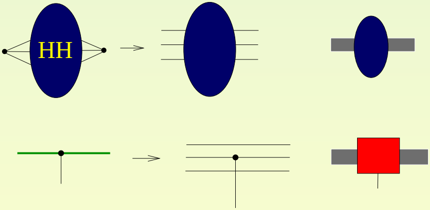

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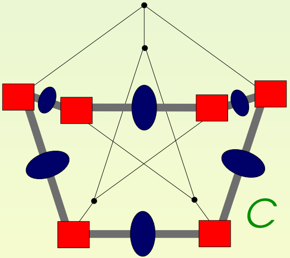

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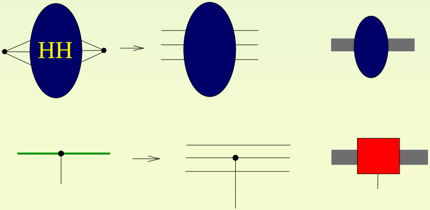

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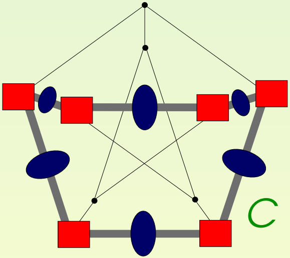

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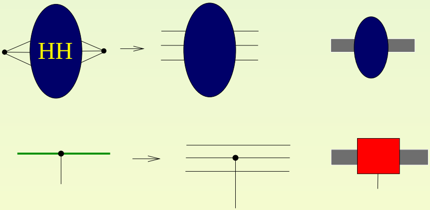

Theorem (Máčajová, Škoviera)

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(I) G^* is NOT hamiltonian;

(II) G^* contains a HH circuit for any $v \in V(G^*)$.

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- If G^* is a snark, then it is not hamiltonian
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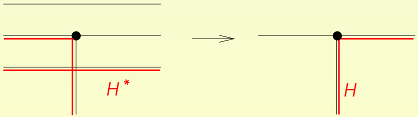

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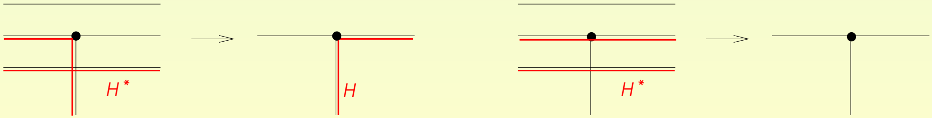

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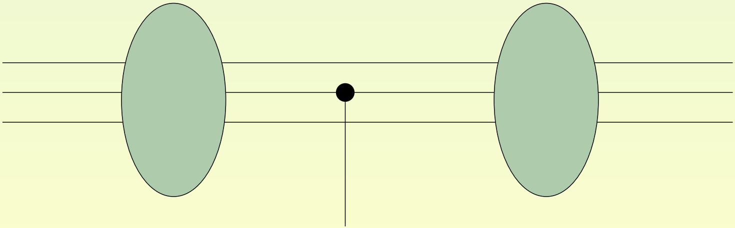

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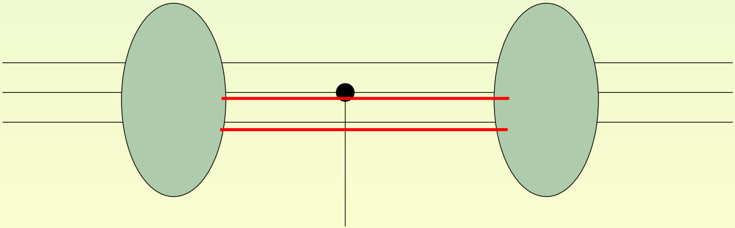

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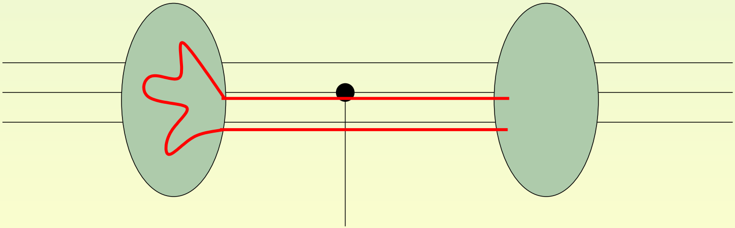

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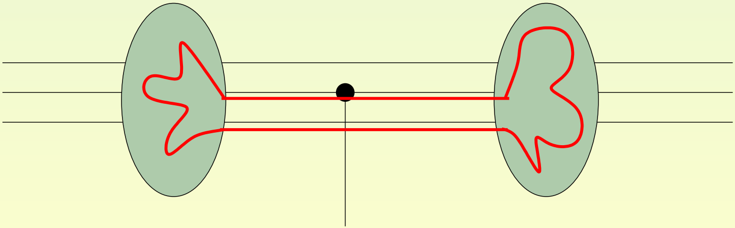


Proof

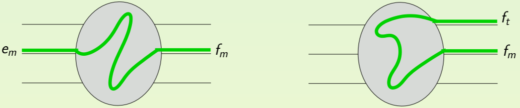
(II) G^* **contains a HH circuit for any** $v \in V(G^*)$

Theorem (Máčajová, Škoviera, 2006)

If edges are replaced by feasible dipoles $\Rightarrow$ G^* *contains a HH circuit for any* $v \in V(G^*)$

Paths required in a feasible dipole

Type O



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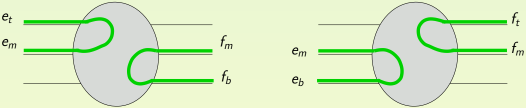

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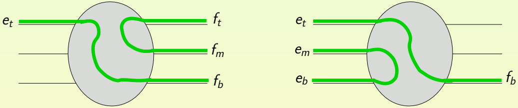

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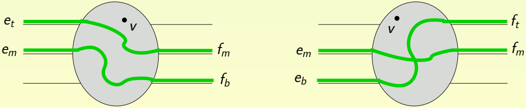
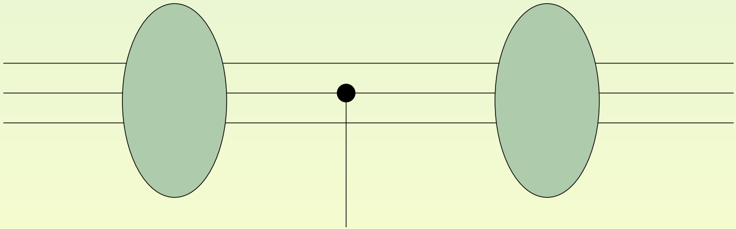


Infinitely many cyclically 7-connected HHCG?



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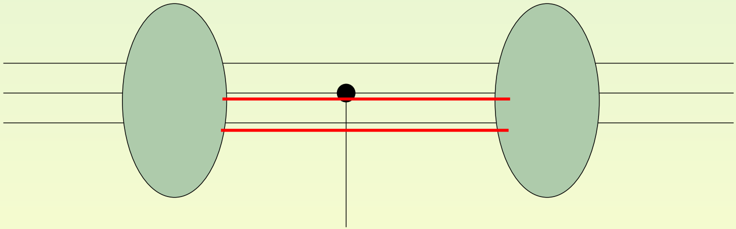

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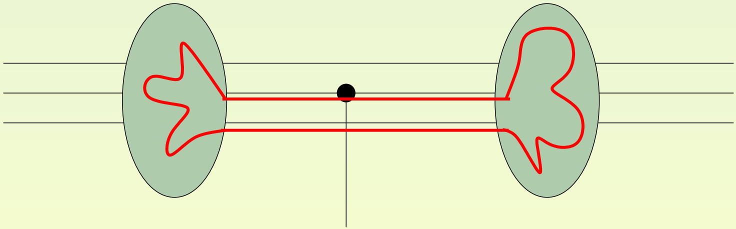

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