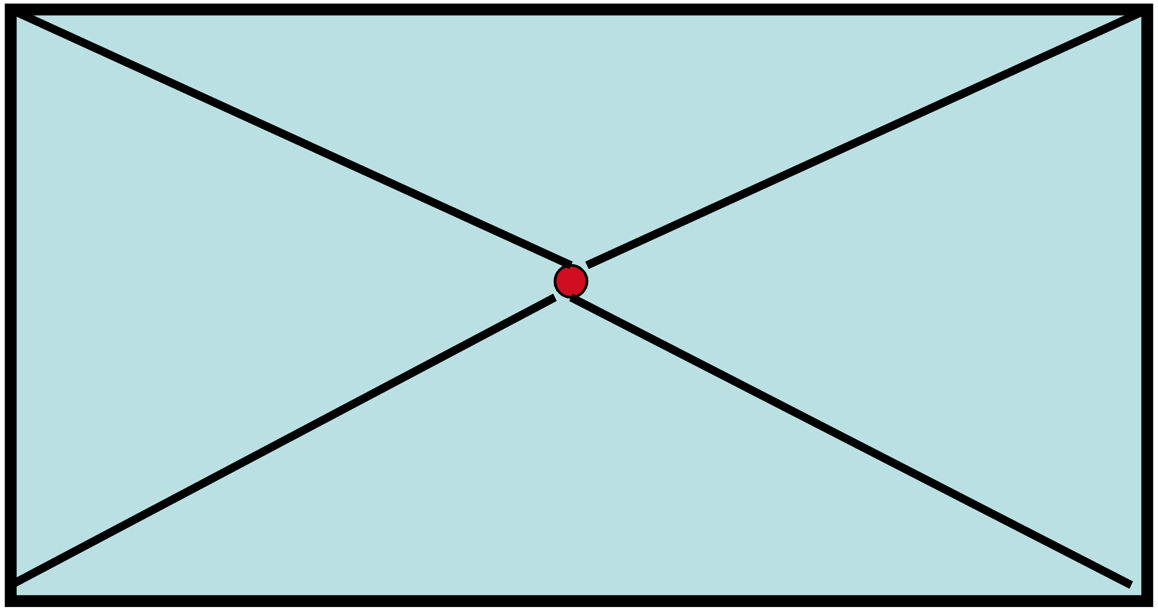
Cubic Subgraphs

Why?

Why not?



5 vertices, 8 edges, claw-free, no cubic subgraph.

- **“Every simple 4-regular graph contains a cubic subgraph”**

- **Berge-Sauer conjecture. (197?)**

- **Zhang Limin, Journal of Changsha Railway Institute, 1 (1985) 130-154**

General questions

1. **How many edges does a graph G of order n must have in order to have a cubic subgraph?**

2. **How many edges does a claw-free graph must have in order to have a cubic subgraph?**

3. **Any other reasonable restrictions?**

Chevalley-Warning Theorem

- Let $P_i(x_1,\ldots,x_n)$ be m polynomials with coefficients in F with characteristic p.
- Let d_i be the total degree of P_i.

- **Theorem: if** $$\sum_{i=1}^{m} d_i < n$$

Then the number of solutions of the system of equations $P_i(x_1,\ldots,x_n) = 0$ is a multiple of p.

Finding cubic subgraphs with Chevalley-Warning.

- **<u>Theorem</u>: Let G be a graph of order G,**
- **size $\geq 2n + 1$, $\Delta(G) \leq 5$ then G contains a cubic subgraph.**

- **Proof: let $E(G) = \{e_1,\ldots,e_k\}$,**
- **$V(G) = \{v_1,\ldots,v_n\}$**

- **Define: $\alpha_{r,s} = 1$ if v_r is incident with e_s and 0 otherwise.**

$$Let \quad P_i(x_1,\ldots,x_m) = \sum_{t=1}^{m} \alpha_{i,t} x_t^2$$

Over GF(3) :
The equations P_i = 0 determine
n polynomial equations in m variables.
Since the degree of each polynomial is 2 and since m > 2n, by C-W theorem the number of solution is 0 mod 3.

- **Since $x_i = 0$ is obviously a solution we must have a non-trivial solution.**

- **Observe that $x_i^2 = 1$ in GF(3) (if $x_i \neq 0$) this means that for each vertex v_i :**

$$\sum_{t=1}^{m} \alpha_{i,t} x_t^2 = 3a$$

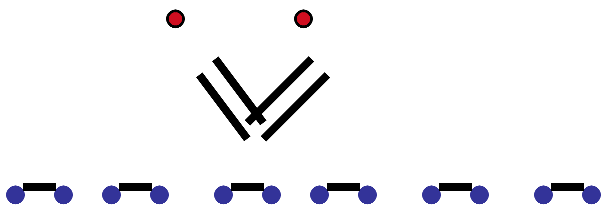
Since deg(v_i) $\leq$ 5, the number of edges incident with v_i is 0 or 3 or:

A CUBIC SUBGRAPH!

Finale

1. **Remark: note that without the restriction $\Delta(G) \leq 5$ we obtain a subgraph with degrees 3k.**
2. **How many edges a graph G of order n must have to guarantee a cubic subgraph?**
3. **Conjecture: 5n/2.**
4. **How many edges a claw-free graph G of order n must have to guarantee a cubic subgraph?**

A graph without cubic subgraphs



A graph with n vertices, 5n/2 - 5 edgeswith no cubic subgraph.