Line graphs of multigraphs and Hamilton-connectedness of claw-free graphs

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Graph: no loops, no parallel edges

Multigraph: no loops, parallel edges possible

$c(G)$: the circumference of G (the length of a longest cycle in G)

$\mathrm{cl}(G)$: the closure of G

Theorem [1997]. *Let G be a claw-free graph. Then*

(*i*) $\mathrm{cl}(G)$ *is uniquely determined,*

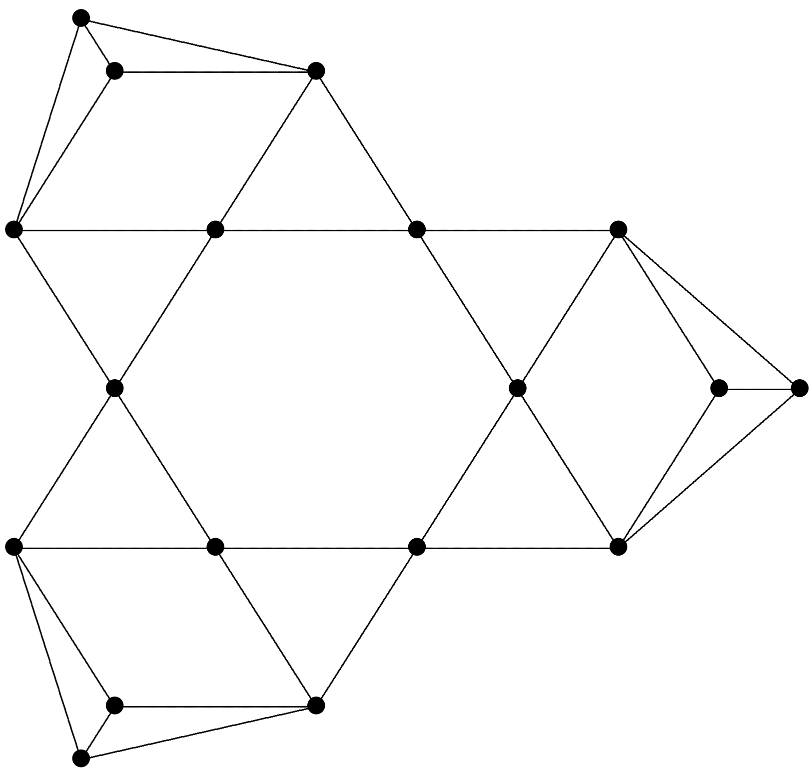
(*ii*) $c(\mathrm{cl}(G)) = c(G)$,

(*iii*) $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

Corollary. *G is hamiltonian if and only if* $\mathrm{cl}(G)$ *is hamiltonian.*

The closure operation $\mathrm{cl}(G)$:

- turns a claw-free graph into a line graph
- preserves hamiltonicity or non-hamiltonicity



An immediate application:

Theorem [Zhan, 1991]. *Every 7-connected line graph (of a multigraph) is Hamilton-connected.*

From this:

Theorem [1997]. *Every 7-connected claw-free graph is hamiltonian.*

Theorem [Hu, Tian, Wei 2005]. *Every 6-connected line graph (of a multigraph) with at most 29 vertices of degree 6 is Hamilton-connected.*

Corollary. *Every 6-connected claw-free graph with at most 29 vertices of degree 6 is hamiltonian.*

Hamilton-connectedness:

Theorem [Brandt, 1999]. *Every 9-connected claw-free graph is Hamilton-connected.*

Theorem [Hu, Tian, Wei, 2005]. *Every 8-connected claw-free graph is Hamilton-connected.*

$\mathcal{C}$ – a class of graphs

We say that $\mathcal{C}$ *is stable* if $\quad G \in \mathcal{C} \Rightarrow \mathrm{cl}(G) \in \mathcal{C}$.

Examples:

k-connected claw-free graphs

chordal claw-free graphs

$\mathcal{P}$ – a property

$\mathcal{C}$ – a stable class

We say that $\mathcal{P}$ *is stable in* $\mathcal{C}$ if, for any $G \in \mathcal{C}$, $\quad G$ has $\mathcal{P} \Leftrightarrow \mathrm{cl}(G)$ has $\mathcal{P}$.

Example:

hamiltonicity is a stable property in the class of k-connected claw-free graphs.

π – a graph invariant

$\mathcal{C}$ – a stable class

We say that π *is stable in* $\mathcal{C}$ if, for any $G \in \mathcal{C}$, $\quad \pi(G) = \pi(\mathrm{cl}(G))$.

Example:

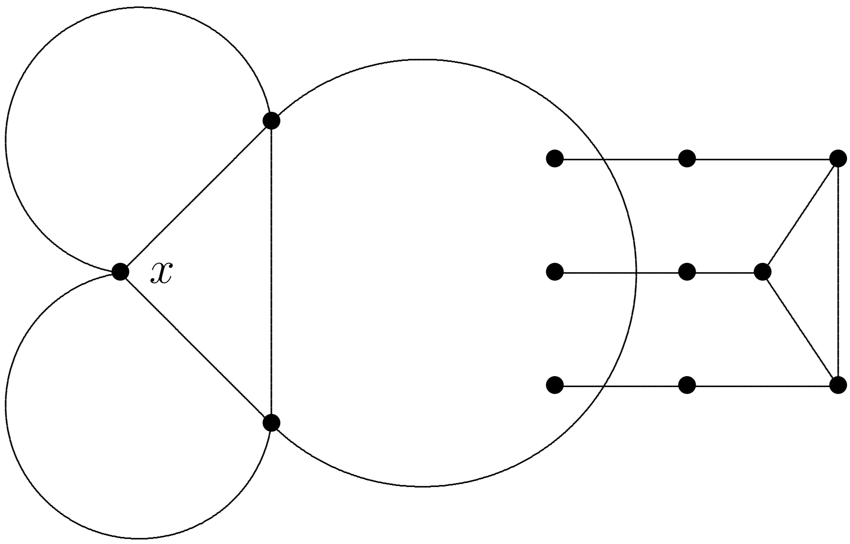
circumference is a stable invariant in the class of k-connected claw-free graphs.

Which properties are stable?

Property / invariant	Stable	Connectivity
Circumference	YES	1
Hamiltonicity	YES	1
Length of a longest path	YES	1
Traceability	YES	1
Homogeneous traceability	NO	3
	???	$4 \leq \kappa \leq 6$
	YES	7
(Vertex) pancyclicity	NO	any $\kappa \geq 2$
(Full) cycle extendability	NO	any $\kappa \geq 2$
Having a 2-factor with $\leq k$ components	YES	1
Minimum number of components in a 2-factor	YES	1
Having a cycle cover with $\leq k$ cycles	YES	1
Minimum number of cycles in a cycle cover	YES	1
Having a path factor with $\leq k$ components	YES	1 [Ishizuka]
Minimum number of components in a path factor	YES	1 [Ishizuka]
Having a path cover with $\leq k$ paths	YES	1 [Ishizuka]
Minimum number of paths in a path cover	YES	1 [Ishizuka]
Hamiltonian index	YES	1
Having hamiltonian prism	YES	1 [Čada]
Having a P_3-factor	NO	1
	YES	2 [Kaneko et al.]
Hamilton-connectedness	NO	3
	???	$4 \leq \kappa \leq 7$
	YES	8 [Hu,Tian,Wei]
Hamilton-connectedness	NO	3
	???	$4 \leq \kappa \leq 6$
	YES	7 [ZR, Vrána,2008]

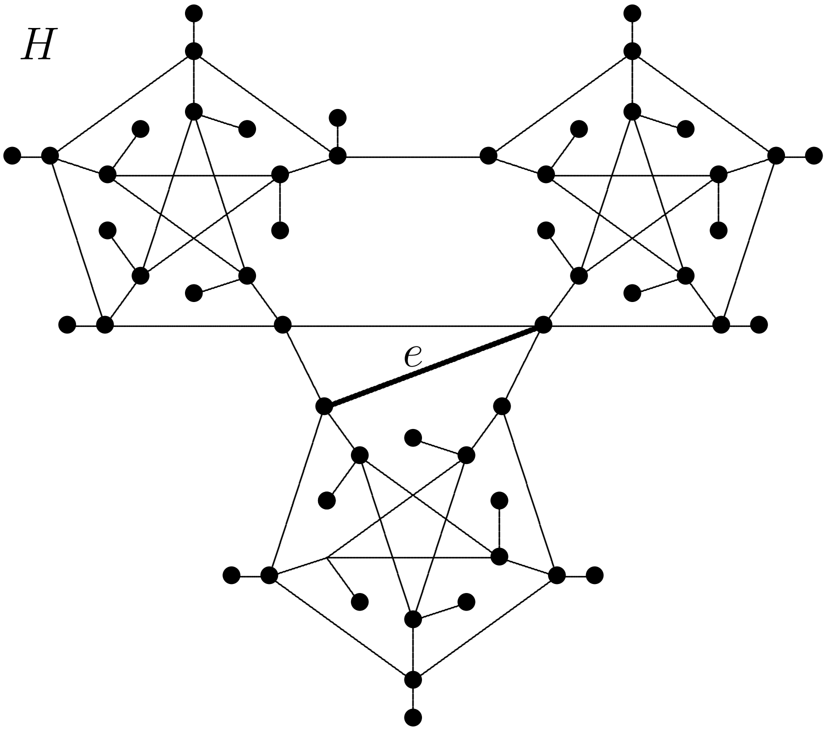
$p_a(G)$ - the length of a longest path with one endvertex at a (given) vertex $a \in V(G)$

Homogeneous traceability (having a hamiltonian path with one endvertex at a for every $a \in V(G)$)



NOT STABLE

$\left.\begin{array}{l}\text{Homogeneous traceability}\\ p_a(G)\end{array}\right\}$ is not stable in 2-conected claw-free graphs.



$G = L(H)$ is 3-connected, not homogeneously traceable

$\mathrm{cl}(G)$ is homogeneously traceable

$\left.\begin{array}{l}\text{Homogeneous traceability} \\ p_a(G)\end{array}\right\}$ is not stable in 3-conected claw-free graphs.

BUT: every 7-connected claw-free graph is hamiltonian $\Rightarrow$

every 7-connected claw-free graph is homogeneously traceable $\Rightarrow$

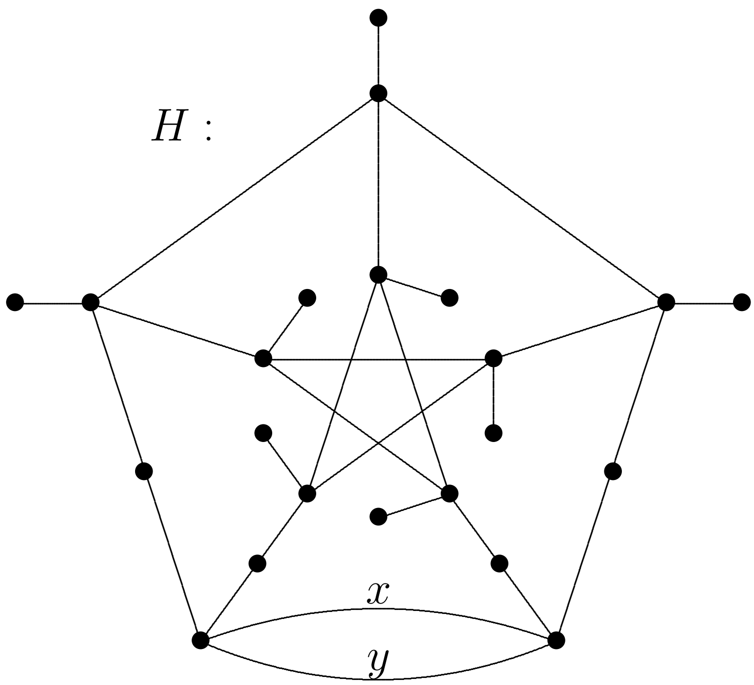
$\left.\begin{array}{l}\text{Homogeneous traceability} \\ p_a(G)\end{array}\right\}$ IS STABLE 7-conected claw-free graphs.

Unstable properties can be made stable by restricting the class of graphs under consideration.

$p_{ab}(G)$ - the length of a longest (a,b)-path (for given $a,b \in V(G)$)

Hamilton-connectedness

$G = L(H)$



$G = L(H)$ is Hamilton-connected $\Longleftrightarrow$ H contains an internally dominating (x,y)-trail for any $x,y \in E(G)$

$G = L(H)$ is 3-connected, not Hamilton-connected

cl(G) is Hamilton-connected

$\left.\begin{array}{l}\text{Hamilton-connectedness}\\ p_{ab}(G)\end{array}\right\}$ is not stable in 3-conected claw-free graphs.

BUT: every 8-connected claw-free graph is Hamilton-connected $\Rightarrow$

$\left.\begin{array}{l}\text{Hamilton-connectedness}\\ p_{ab}(G)\end{array}\right\}$ IS STABLE in 8-conected claw-free graphs.

k-closure

$x \in V(G)$ is *locally k-connected* if $N(x)$ induces a k-connected graph.

$\mathrm{cl}_k(G)$: local completions only at locally k-connected vertices.

Theorem [Bollobás, Riordan, ZR., Saito, Schelp, 1999].
(*i*) $\mathrm{cl}_k(G)$ *is uniquely determined for every* $k \geq 1$.
(*ii*) *Homogeneous traceability is stable under* $\mathrm{cl}_2(G)$.
(*iii*) *Hamilton-connectedness is stable under* $\mathrm{cl}_3(G)$.

Conjecture [Bollobás, Riordan, ZR., Saito, Schelp, 1999].
Hamilton-connectedness is stable under $\mathrm{cl}_2(G)$.

A. Kelmans: *On graph closures.* **Discrete Math.** 271 (2003), 141-168

Claims to give infinitely many counterexamples to the conjecture

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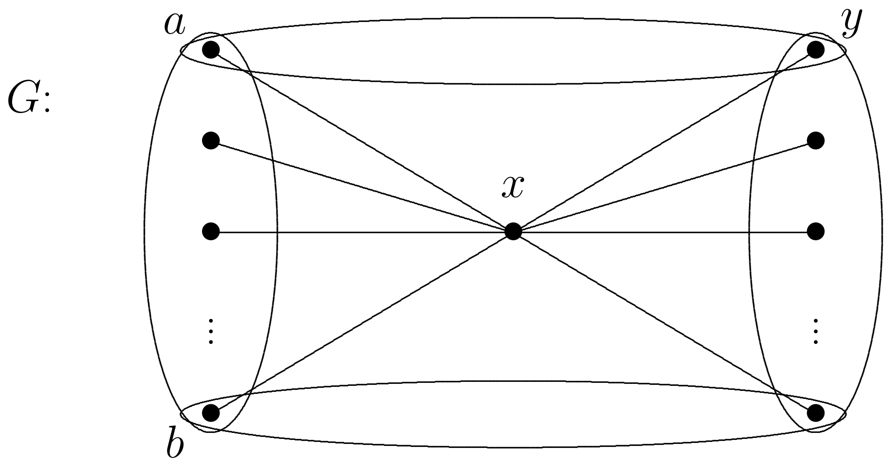
WRONG

Example.



- $\langle N(x) \rangle$ 2-connected
- No hamiltonian (a, b)-path
- There is a hamiltonian (a, b)-path in G'_x.

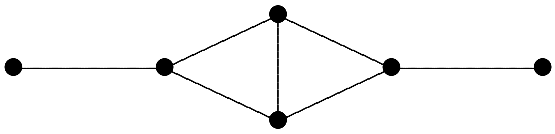
"Having a hamiltonian (a, b)-path" / $p_{ab}(G)$ } is not stable under $\mathrm{cl}_2(G)$.

BUT: G'_x has no hamiltonian (a, y)-path.
Thus: G'_x is NOT Hamilton-connected.
The example does NOT disprove the conjecture.

Theorem [ZR., Vrána, 2008].
Hamilton-connectedness is stable under $\mathrm{cl}_2(G)$.

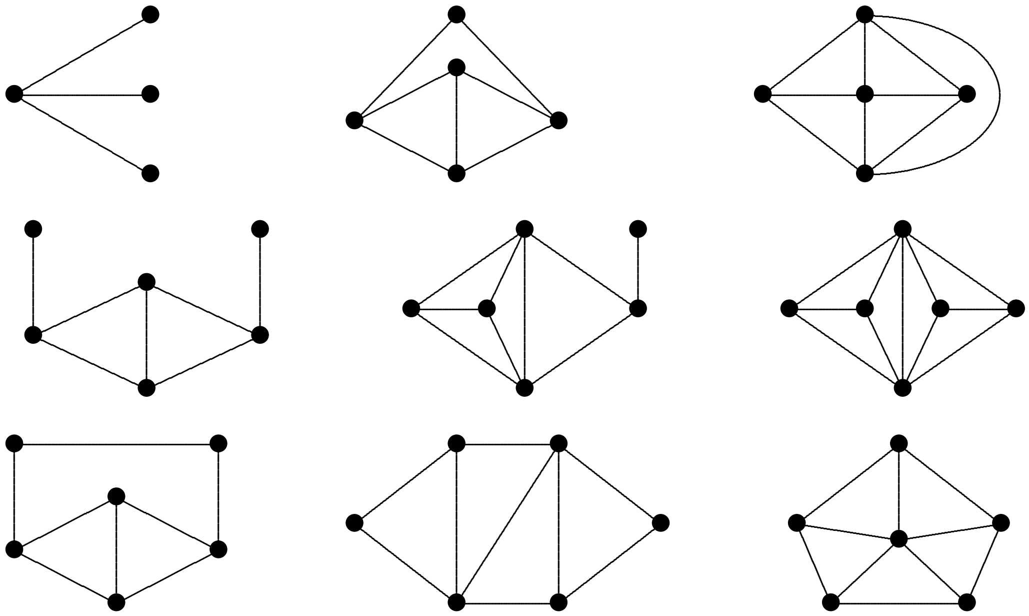
G is *2-closed* if $G = \mathrm{cl}_2(G)$

What is the structure of 2-closed graphs?

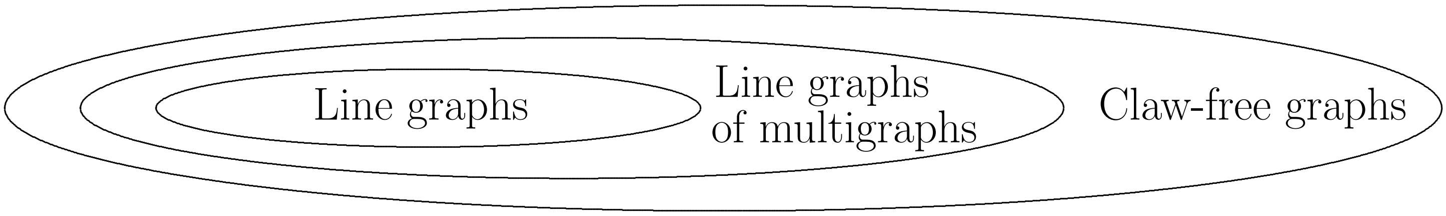
Not a line graph: 

Theorem [Beineke, 1969].

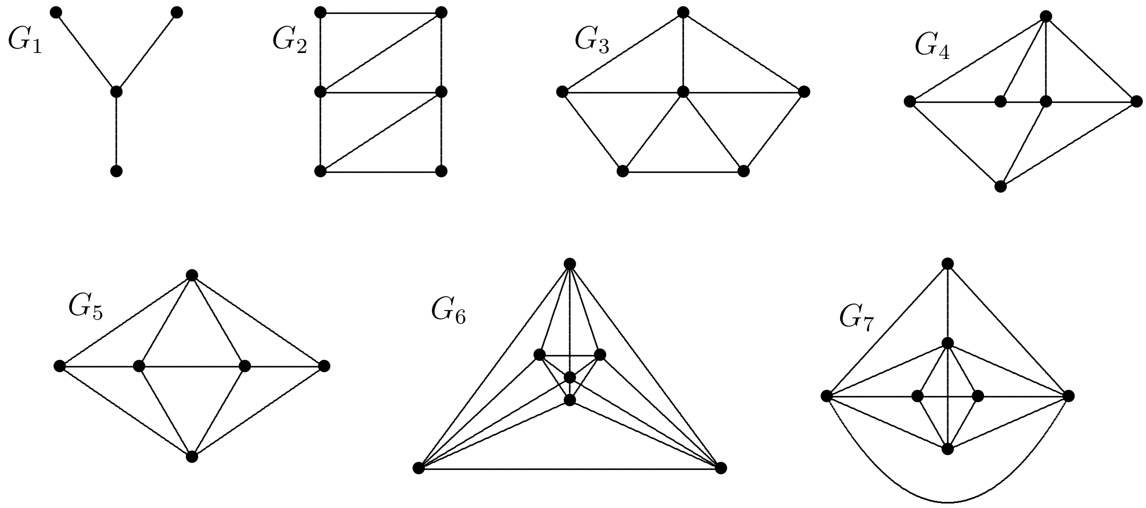
A graph G is a line graph (of some graph) if and only if G does not contain a copy of any of the following graphs as an induced *subgraph.*



Line graph of a multigraph?



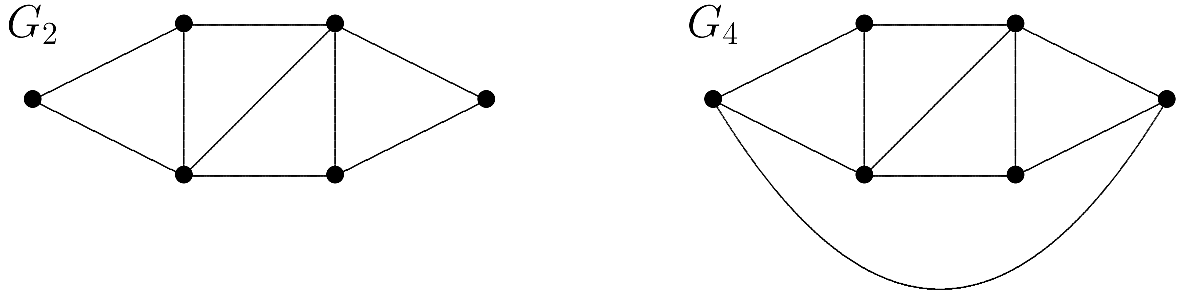
Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph G is a line graph of a multigraph *if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



Lemma. *Let G be a claw-free graph, $x \in V(G)$, let $H \overset{\scriptscriptstyle IND}{\subset} \langle N(x) \rangle_G$ be a 2-connected graph containing two distinct pairs of independent vertices. Then $\langle N(x) \rangle_G$ is 2-connected.*

Lemma. *Every 2-closed claw-free graph is $\{G_1, G_3, G_5, G_6, G_7\}$-free.*

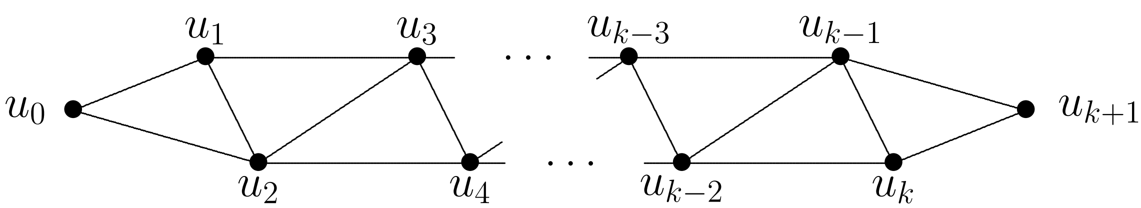
Thus, in $\mathrm{cl}_2(G)$, only G_2 or G_4 can remain.



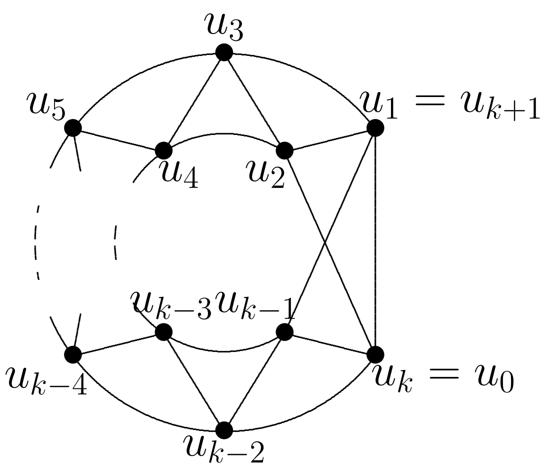
"Multigraph closure" $\mathrm{cl}^M(G)$ of a graph G:
Recursive performing $\mathrm{cl}_2(G)$ and closing specified vertices in copies of G_2 or G_4, as long as there is something to do.

$J = u_0 u_1 \ldots u_{k+1}$ – a walk in G. We say that J is *good in* G, if

- $k \geq 4$,
- $J^2 \subset G$,
- for any i, $0 \leq i \leq k-4$, $\langle \{u_i, u_{i+1}, \ldots, u_{i+5}\} \rangle_G \simeq S_1$ or S_2.



Similarly – a *good cycle*



Lemma. *Let G be a 2-closed claw-free graph and $J = u_0u_1 \dots u_{k+1}$ a good walk in G, $k \geq 5$. Then $d_G(u_i) = 4$, $i = 3, \dots, k-2$.*

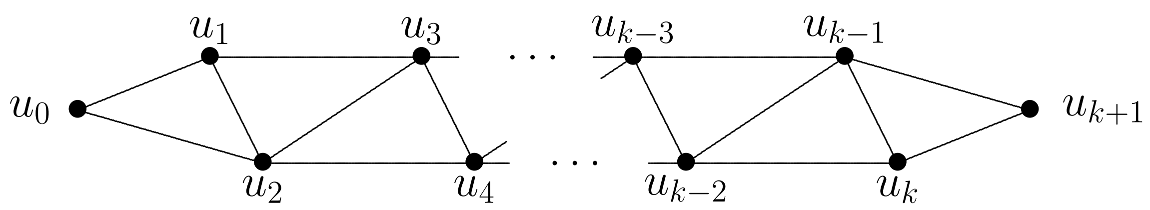
Lemma. *Let G be a connected 2-closed claw-free graph and let $C \subset G$ be a good cycle in G, $|V(C)| \geq 6$. Then $G = C^2$.*

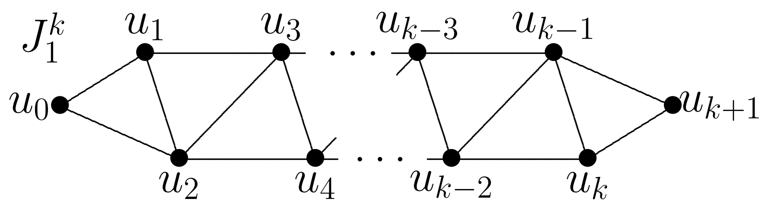
In the rest we suppose that G is not the square of a cycle.

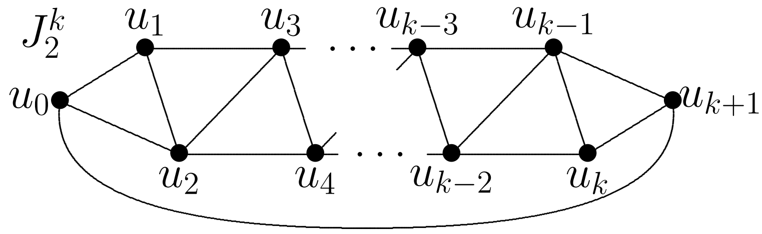
Let J be a good walk in G. We say that J is *maximal* if, for every good walk J' in G, J being a subsequence of J' implies $J = J'$.

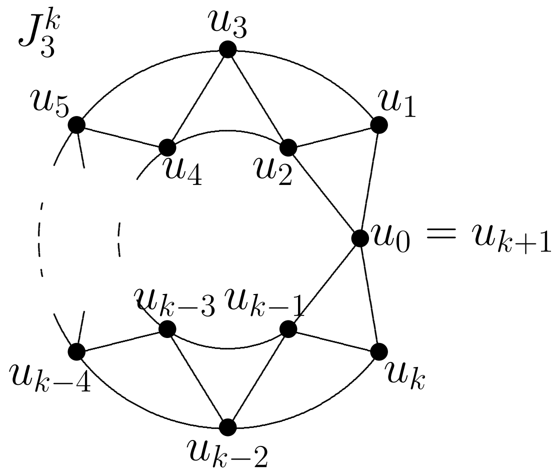
Lemma. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, and let $J = u_0, u_1, \dots, u_{k+1}$ be a maximal good walk in G. Then $\langle N_G[u_1] \setminus \{u_3\}\rangle_G = \langle N_G[u_2] \setminus \{u_3, u_4\}\rangle_G$ and this subgraph is a clique.*

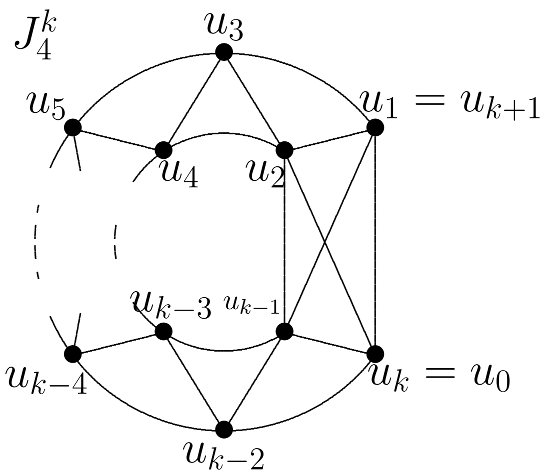
Lemma. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, and let $J = u_0u_1 \dots u_{k+1}$ be a good walk in G. Then $u_1 \dots u_k$ is a path.*




J_1^k
u_0 u_1 u_2 u_3 u_4 u_{k-3} u_{k-2} u_{k-1} u_k u_{k+1}


J_2^k
u_0 u_1 u_2 u_3 u_4 u_{k-3} u_{k-2} u_{k-1} u_k u_{k+1}


J_3^k
u_3 u_5 u_1 u_4 u_2
$u_0 = u_{k+1}$
u_{k-3} u_{k-1} u_{k-4} u_k u_{k-2}


J_4^k
u_3 u_5 $u_1 = u_{k+1}$ u_4 u_2
u_{k-3} u_{k-1}
u_{k-4} $u_k = u_0$ u_{k-2}

Let G be a claw-free graph, and let $\mathrm{cl}^M(G)$ be a graph constructed by the following algorithm.

1. Set $G_1 = \mathrm{cl}_2(G)$, $i := 1$.
2. If G_i contains a good walk, then
   (a) choose a maximal good walk $J = u_0u_1 \ldots_{k+1}$,
   (b) let G_i' be the local completion of G_i at u_1 and G_i'' be the local completion of G_i' at u_k,
   (c) set $G_{i+1} = \mathrm{cl}_2(G_i'')$, $i := i + 1$,
   (d) go to (2).
3. Set $\mathrm{cl}^M(G) = G_i$.

The graph $\mathrm{cl}^M(G)$ will be called (the) *M-closure* of G.

Theorem. *Let G be a claw-free graph. Then*
(*i*) $\mathrm{cl}^M(G)$ *is uniquely determined,*
(*ii*) *there is a multigraph H such that* $\mathrm{cl}^M(G) = L(H)$,
(*iii*) *G is Hamilton-connected if and only if* $\mathrm{cl}^M(G)$ *is Hamilton-connected.*

Hamilton-connectedness is stable under $\mathrm{cl}^M(G)$.

BUT:

"Having a hamiltonian (a, b)-path"
 $p_{ab}(G)$ } is NOT STABLE under $\mathrm{cl}^M(G)$!!!

Proposition.

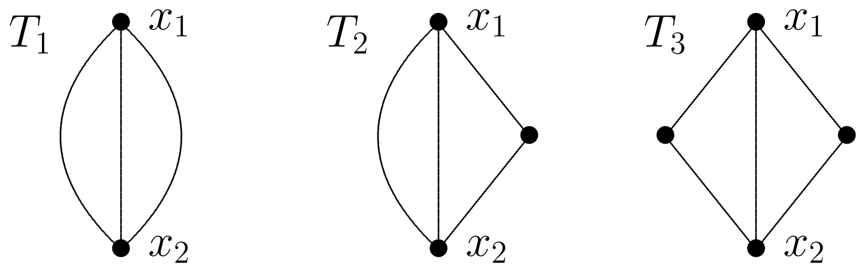
A claw-free graph G is M-closed if and only if there is a multigraph H such that $G = L(H)$ and H does not contain a subgraph S (not necessarily induced) with any of the following properties:

(i) $S \simeq T_1$ *and there are* $u_1, u_2 \in V(H) \setminus V(S)$ *such that* $u_1 \neq u_2$ *and* $u_i x_i \in E(H)$, $i = 1, 2$,

(ii) $S \simeq T_2$ *or* $S \simeq T_3$ *and there is a* $u \in V(H) \setminus V(S)$ *such that* $|N_H(u) \cap \{x_1, x_2\}| = 1$,

(iii) $S \simeq T_3$

(where x_1, x_2 *are the only vertices in* S *with* $d_S(x_i) = 3$*).*



Recall: Every 6-connected line graph (of a multigraph) with at most 29 vertices of degree 6 is Hamilton-connected [Hu, Tian, Wei 2005].

Theorem. *Every 6-connected claw-free graph with at most 29 vertices of degree 6 is Hamilton-connected.*

Corollary. *Every 7-connected claw-free graph is Hamilton-connected.*

Conjecture A [Matthews, Sumner, 1984].
Every 4-connected claw-free graph is hamiltonian.

Conjecture B.
Every 4-connected line graph of a multigraph is Hamilton-connected.

Conjecture C.
Every 4-connected claw-free graph is Hamilton-connected.

Theorem [Kužel, Vrána, Xiong].
Conjectures A and B are equivalent.

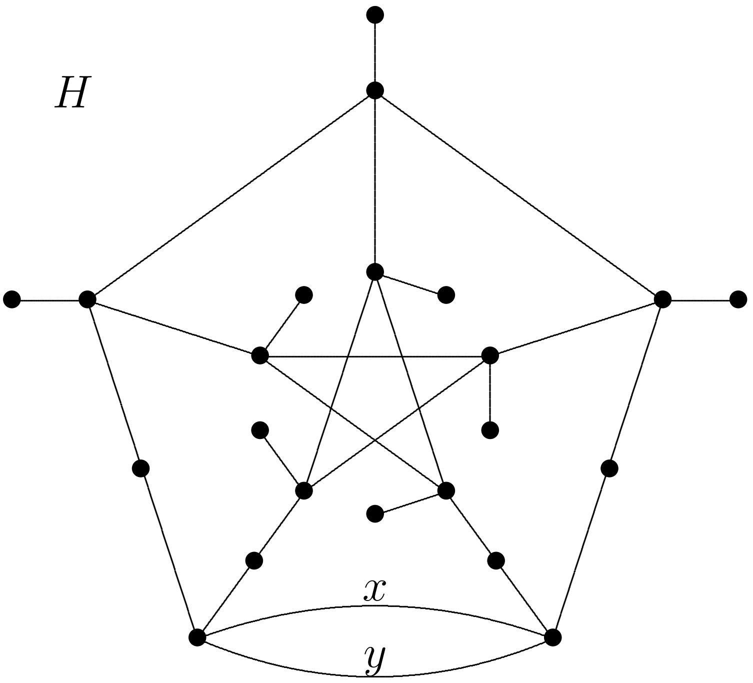
Theorem.
Conjectures B and C are equivalent.

$\mathcal{C}$ – *a class of graphs*

A closure *on* $\mathcal{C}$ *is a mapping* $\mathrm{cl} : \mathcal{C} \to \mathcal{C}$ *such that*

- $V(G) = V(\mathrm{cl}(G))$
- $E(G) \subset E(\mathrm{cl}(G))$

$G = L(H)$:



$\mathcal{L}_k$ – *the class of k-connected line graphs (of graphs)*
$\mathcal{L}_k^M$ – *the class of k-connected line graphs of multigraphs*

Theorem [Vrána, 2008].
There is no closure cl *on* $\mathcal{L}_3^M$ *such that* $\mathrm{cl} : \mathcal{L}_3^M \to \mathcal{L}_3$ *and Hamilton-connectedness is stable under* cl.