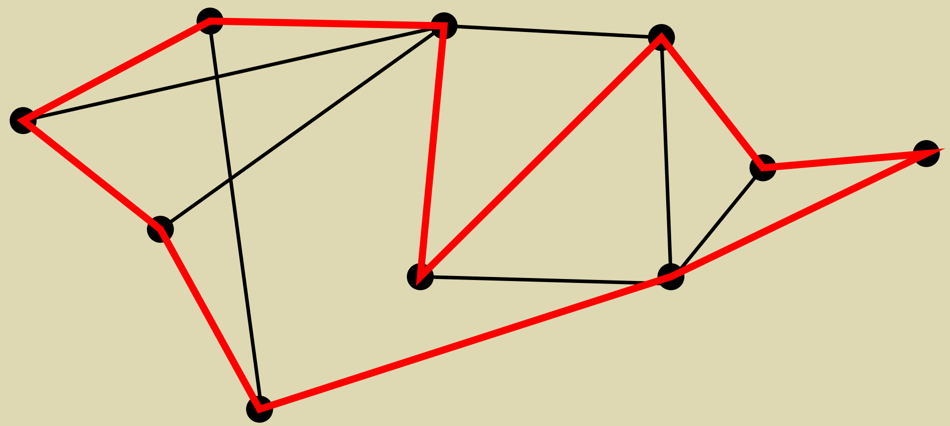
Generalized Hamiltonian Cycles

Jakub Teska

School of ITMS
University of Ballarat, VIC 3353, Australia

Hamiltonian cycle

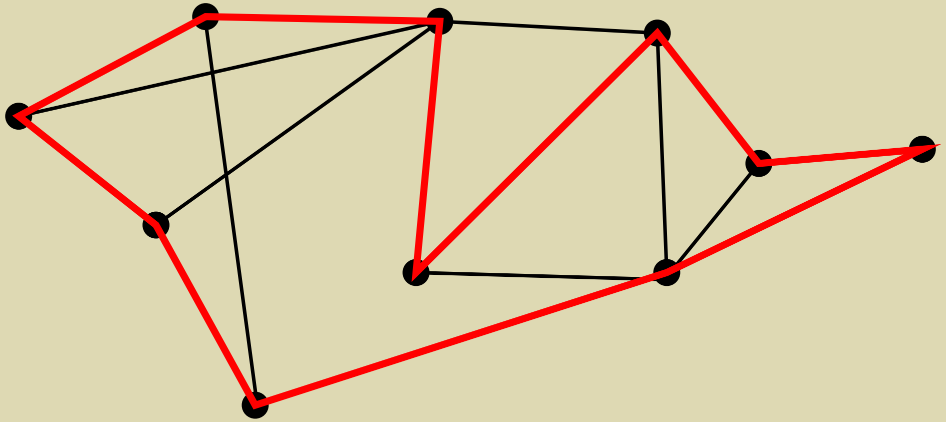
- Hamiltonian cycle is a cycle in a graph which visits every vertex of the graph.



- Decide whether a graph is hamiltonian is well known NP-Complete problem.

Hamiltonian cycle

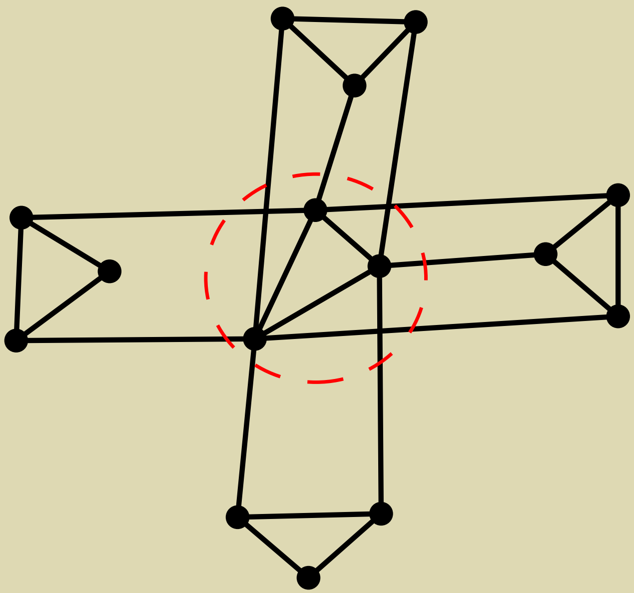
- Hamiltonian cycle is a cycle in a graph which visits every vertex of the graph.



- Decide whether a graph is hamiltonian is well known NP-Complete problem.
- If a graph G is hamiltonian then G is 2-connected.

Toughness

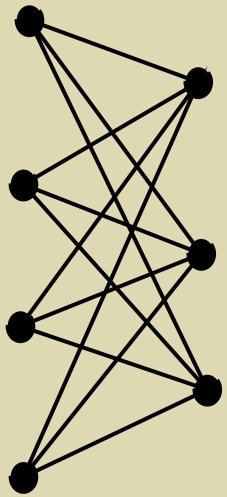
- The toughness of a non-complete graph is $t(G) = min(\frac{|S|}{c(G-S)})$, where the minimum is to be taken over all nonempty vertex sets S, for which $c(G-S) \geq 2$.



Toughness

- If a graph G is t-tough then G is $\lceil 2t \rceil$-connected.

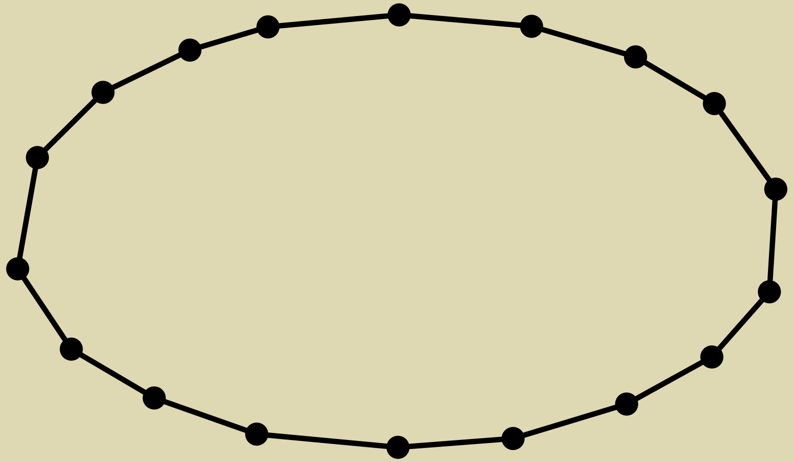
  Opposite implication is not true. There exist graphs with arbitrary large connectivity and arbitrary small toughness.



$K_{m,n}$ for $m \geq n$ is n-connected but toughness $t(K_{m,n}) = \frac{n}{m}$

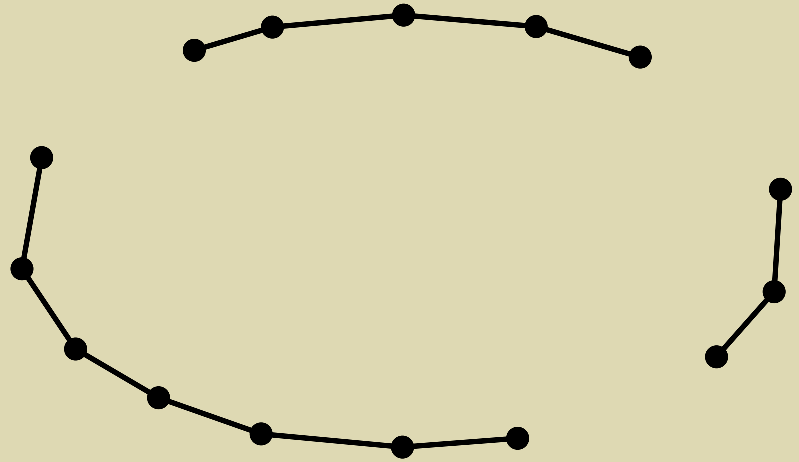
Necessary conditions

- If a graph G is Hamiltonian then G is 1-tough



Necessary conditions

- If a graph G is Hamiltonian then G is 1-tough



- If toughness $t(G) < 1$ then G has no Hamiltonian cycle

Sufficient conditions

Chvátal's Conjecture : There exists a finite constant t_0 such that every t_0-tough graph is hamiltonian.

For many years the focus was on determining whether all 2-tough graphs are hamiltonian. But in 2000 Bauer, Broersma and Veldman proved the following theorem.

Sufficient conditions

Chvátal's Conjecture : There exists a finite constant t_0 such that every t_0-tough graph is hamiltonian.

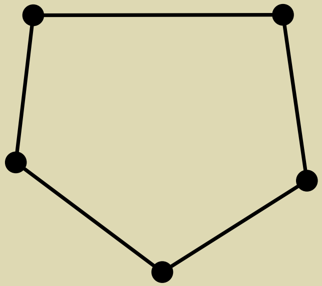
For many years the focus was on determining whether all 2-tough graphs are hamiltonian. But in 2000 Bauer, Broersma and Veldman proved the following theorem.

- For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough graph without a Hamiltonian cycle.

To prove similar theorem to the Chvátal's Conjecture we have to restrict our focus on some special classes of graphs.

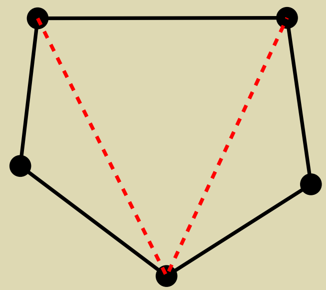
Chordal graphs

- Graph is chordal if every cycle of length greater then three has a chord.



Chordal graphs

- Graph is chordal if every cycle of length greater then three has a chord.



- Vertex x is simplicial vertex in G if $\langle N_G(x)\rangle_G$ is complete graph.

- Assume that graph G is chordal. Then G has a simplicial vertex v and $G - v$ is chordal graph.

  Every chordal graph can be constructed from K_3 just by recursive adding of new simplicial vertices.

Chordal graphs

- Every 18-tough chordal graph is Hamiltonian. (Chen et. al. 1997)
- For every $\epsilon > 0$, there exists a $(\frac{7}{4} - \epsilon)$-tough chordal graph without a Hamiltonian cycle.(Bauer, Broersma and Veldman, 2000)

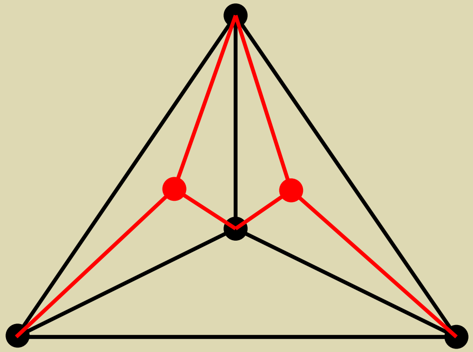
Chordal graphs

- Every 18-tough chordal graph is Hamiltonian. (Chen et. al. 1997)
- For every $\epsilon > 0$, there exists a $(\frac{7}{4} - \epsilon)$-tough chordal graph without a Hamiltonian cycle.(Bauer, Broersma and Veldman, 2000)
- Every chordal planar graph with $t(G) > 1$ is hamiltonian. (Bőhme et. al. 1999)
- There exists a sequence $G_1, G_2, ...$ of 1-tough chordal planar graphs with $\frac{c(G_i)}{|V(G_i)|} \rightarrow 0$ as $i \rightarrow \infty$.

Sketch of the proof

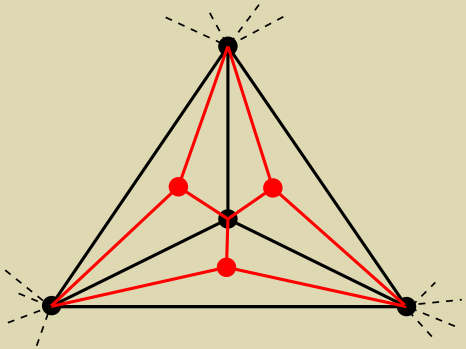
- If $t(G) > 1$ then G is 3-connected. Then degree of every vertex is at least three.
- If G is chordal planar graph, then G does not contain K_5 as a subgraph and therefor degree of every simplicial vertex is at most three.

G can be constructed from K_3 just by recursive adding of new simplicial vertices, but we can do it as follows: In every step we add set S of all simplicial vertices into the neighborhood of a simplicial vertex.



Sketch of the proof

- $|S| < 3$



Sketch of the proof

- $|S| < 3$

  Suppose that from graph G_i we get graph G_{i+1} by adding set S of all simplicial vertices into the neibourhood of a simplicial vertex.

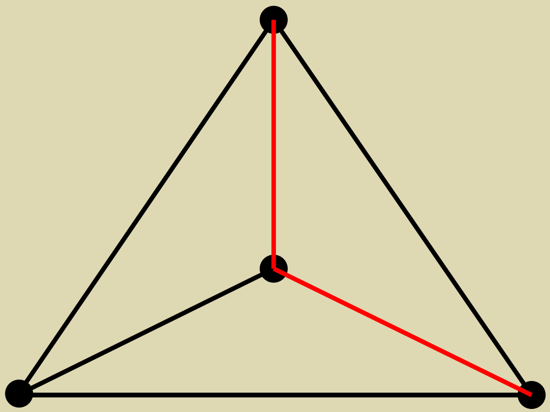
- If G_i is hamiltonian then G_{i+1} is hamiltonian.

Sketch of the proof

- $|S| < 3$

  Suppose that from graph G_i we get graph G_{i+1} by adding set S of all simplicial vertices into the neibourhood of a simplicial vertex.

- If G_i is hamiltonian then G_{i+1} is hamiltonian.

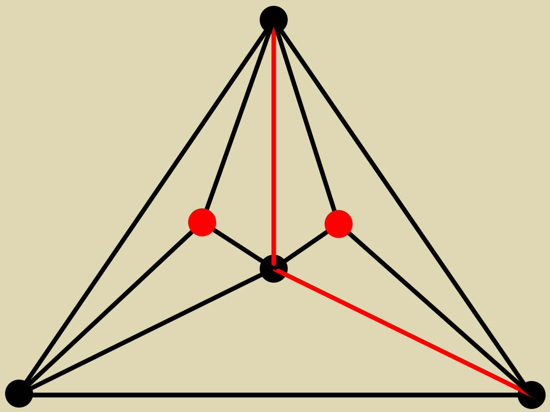


Sketch of the proof

- $|S| < 3$

  Suppose that from graph G_i we get graph G_{i+1} by adding set S of all simplicial vertices into the neibourhood of a simplicial vertex.

- If G_i is hamiltonian then G_{i+1} is hamiltonian.

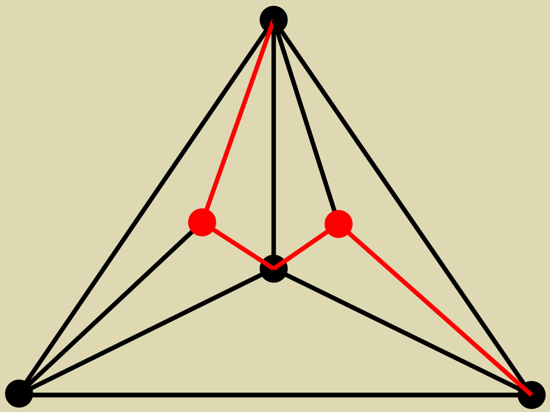


Sketch of the proof

- $|S| < 3$

  Suppose that from graph G_i we get graph G_{i+1} by adding set S of all simplicial vertices into the neibourhood of a simplicial vertex.

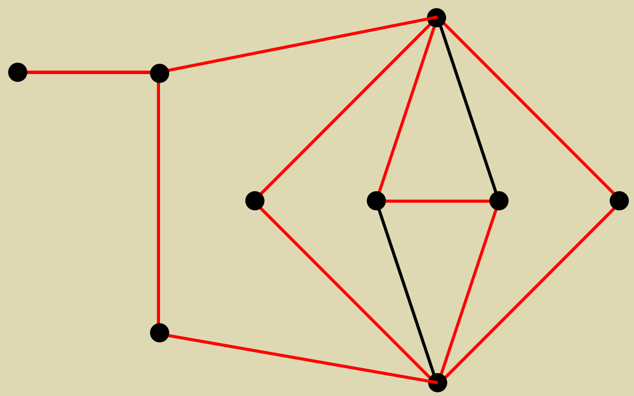
- If G_i is hamiltonian then G_{i+1} is hamiltonian.



k-walks

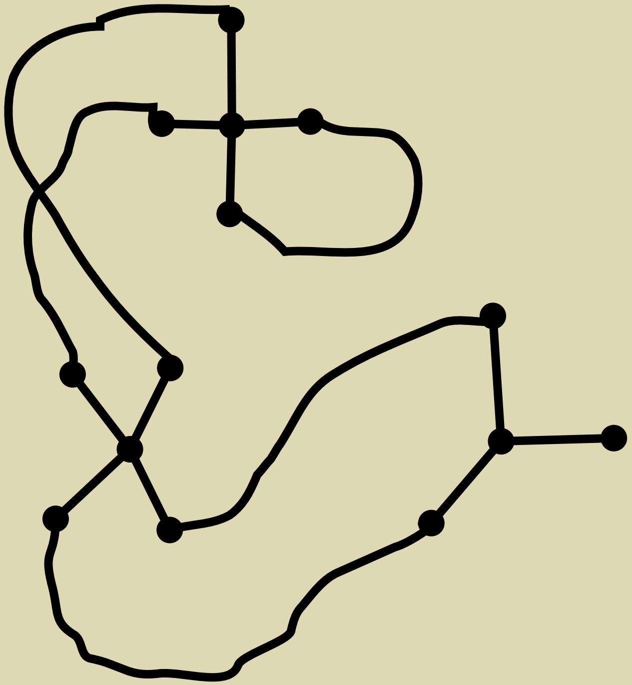
- A k-walk in a graph G is a spanning closed walk which visits every vertex of G at most k-times.

  This generalizes the notion of a Hamiltonian cycle because 1-walk in G is exactly a Hamiltonian cycle in G.



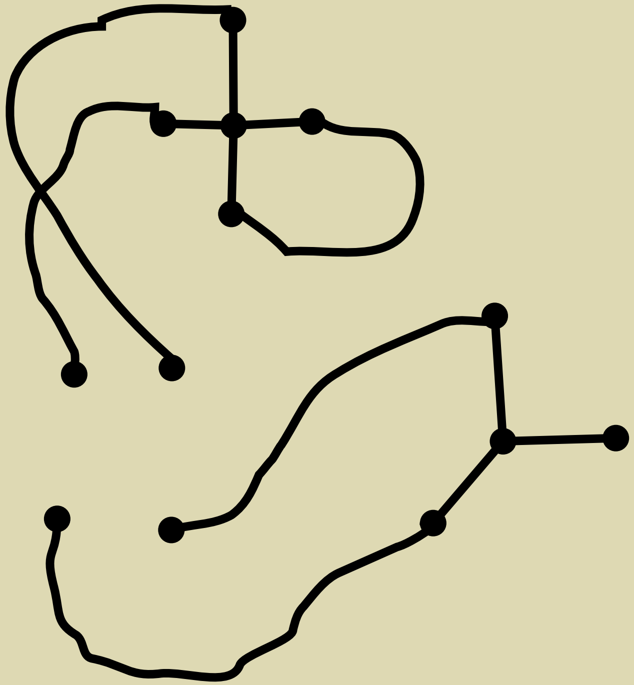
k-walks

- Every graph containing a k-walk is $\frac{1}{k}$-tough.



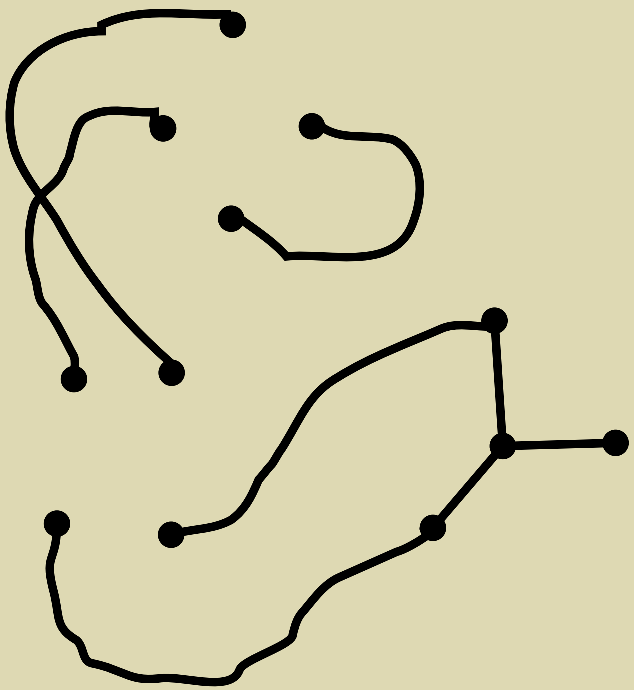
k-walks

- Every graph containing a k-walk is $\frac{1}{k}$-tough.



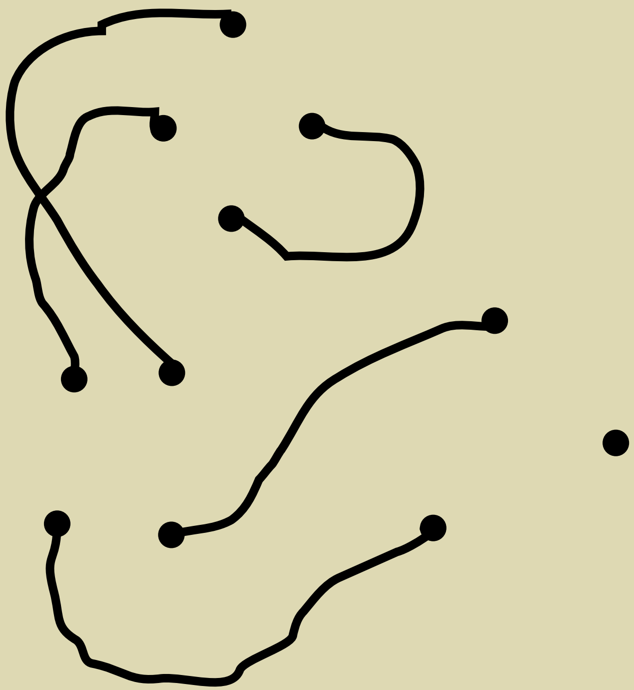
k-walks

- Every graph containing a k-walk is $\frac{1}{k}$-tough.



k-walks

- Every graph containing a k-walk is $\frac{1}{k}$-tough.



If $t(G) < \frac{1}{k}$ then G does not contain a k-walk.

2-walks

- Every 4-tough graph has a 2-walk. (Ellingham, Zha 2000)

  This is similar theorem to the Chvátal's Conjecture for 2-walks

2-walks

- Every 4-tough graph has a 2-walk. (Ellingham, Zha 2000)

  This is similar theorem to the Chvátal's Conjecture for 2-walks

- For every $\epsilon > 0$ and every $k \geq 1$, there exists a $(\frac{8k+1}{4k(2k-1)} - \epsilon)$-tough graph with no k-walk.

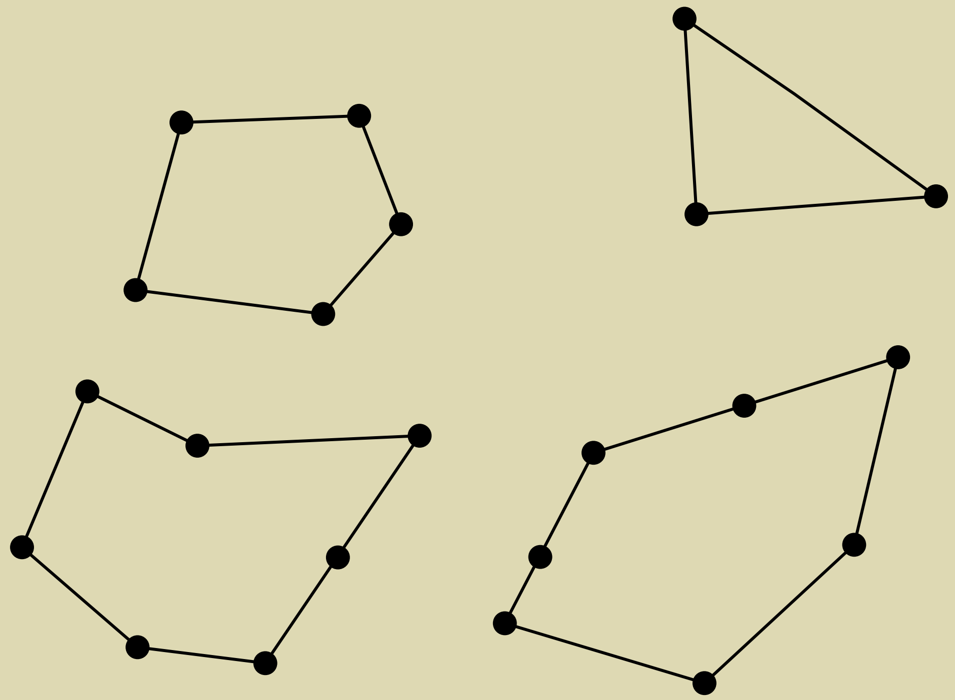
  For $k = 2$ we get that there exists $(\frac{17}{24} - \epsilon)$-tough graph with no 2-walk.

Idea of the proof

- Every 4-tough graph has a 2-walk. (Ellingham, Zha 2000)
- **If G is 2-tough then G has a 2-factor.**

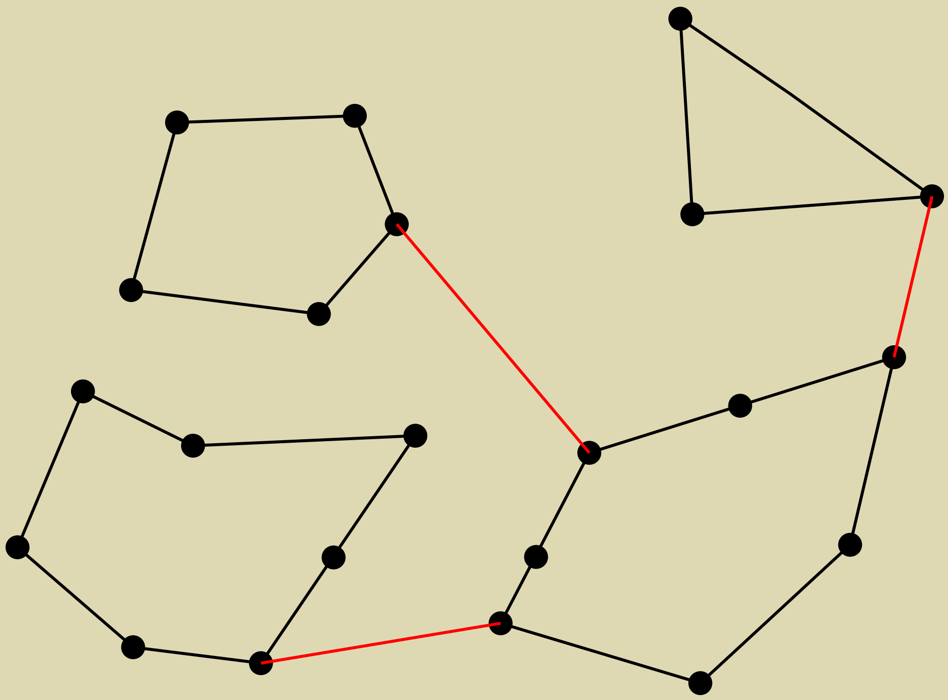
Idea of the proof

- Every 4-tough graph has a 2-walk. (Ellingham, Zha 2000)
- **If G is 2-tough then G has a 2-factor.**



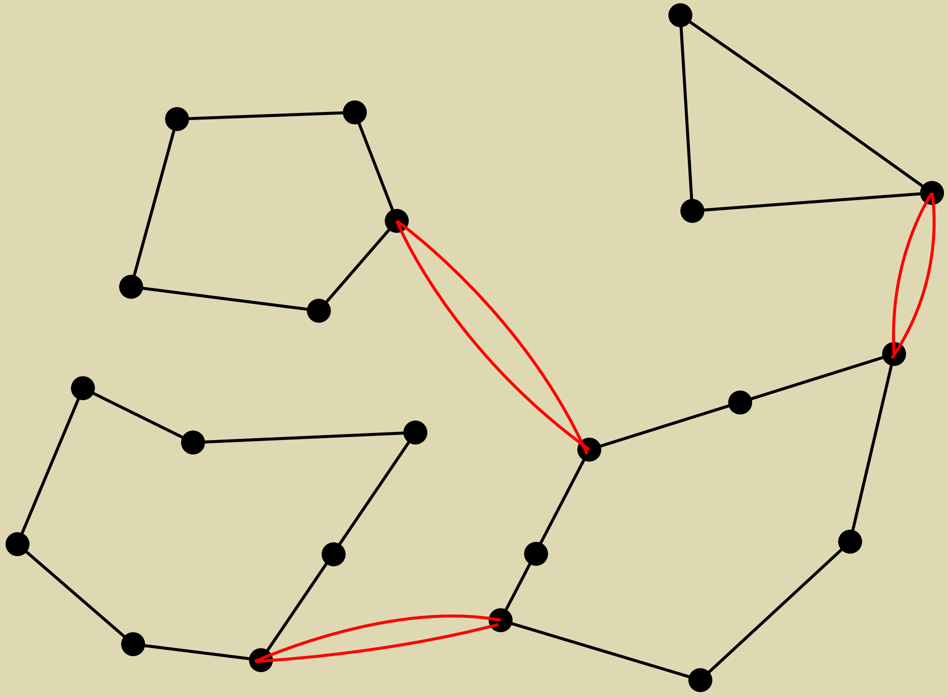
Idea of the proof

- Every 4-tough graph has a 2-walk. (Ellingham, Zha 2000)
- **If G is 2-tough then G has a 2-factor.**



Idea of the proof

- Every 4-tough graph has a 2-walk. (Ellingham, Zha 2000)
- **If G is 2-tough then G has a 2-factor.**



Then Eulerian cycle in this graph coresponds to a 2-walk in the original graph.

New result

- Theorem : Every chordal planar graph with $t(G) > \frac{3}{4}$ has a 2-walk.

New result

- Theorem : Every chordal planar graph with $t(G) > \frac{3}{4}$ has a 2-walk.

  Every simplicial vertex has degree 2 or 3.

New result

- Theorem : Every chordal planar graph with $t(G) > \frac{3}{4}$ has a 2-walk.

  Every simplicial vertex has degree 2 or 3.

  G can be constructed from K_3 just by recursive adding of new simplicial vertices.

  From graph G_i we get graph G_{i+1} by adding set S of all simplicial vertices into the neibourhood of a simplicial vertex.

New result

- Theorem : Every chordal planar graph with $t(G) > \frac{3}{4}$ has a 2-walk.

  Every simplicial vertex has degree 2 or 3.

  G can be constructed from K_3 just by recursive adding of new simplicial vertices.

  From graph G_i we get graph G_{i+1} by adding set S of all simplicial vertices into the neibourhood of a simplicial vertex.

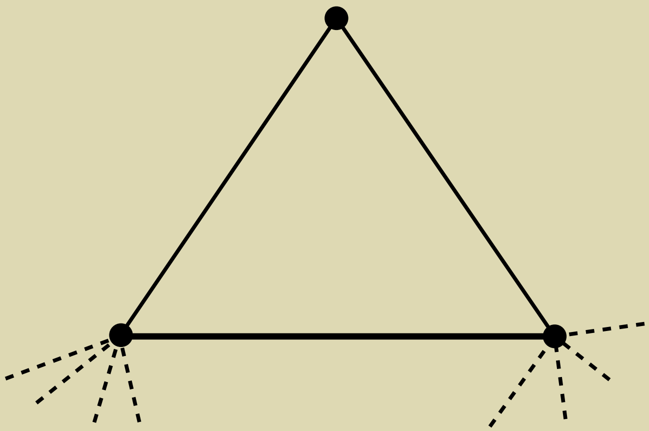
  Now we have the following cases:

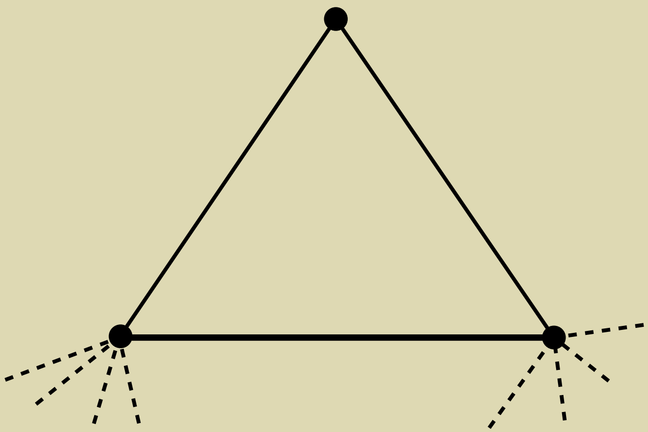
  I) Degree of x in G_i is two
  II) Degree of x in G_i is three

  A) T visits two edges incident with x in G_i
  B) T visits one edge incident with x in G_i

Case I

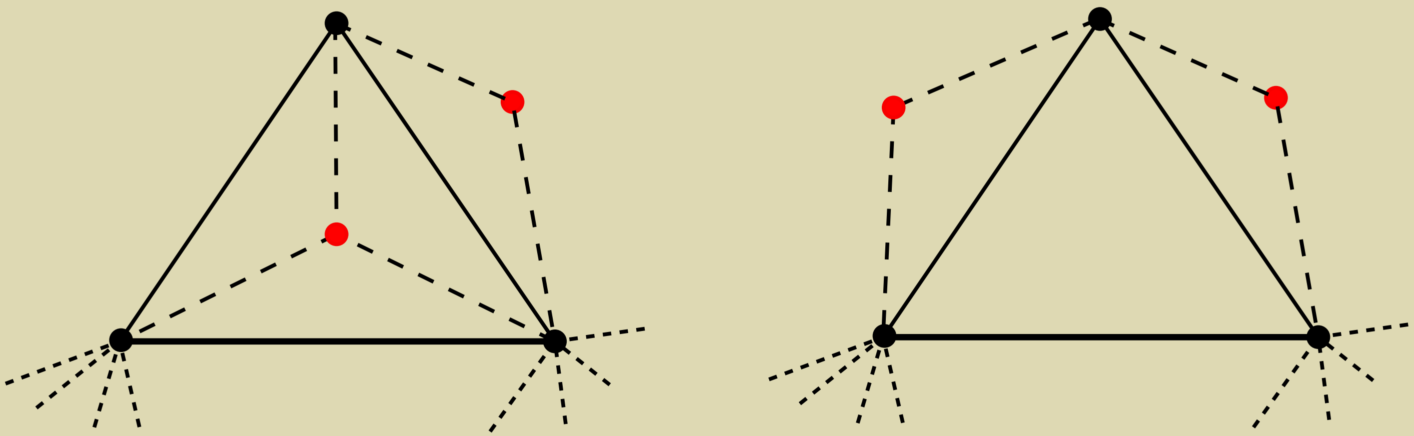
- If degree of x in G_i is two then $|S| \leq 2$.





Case I

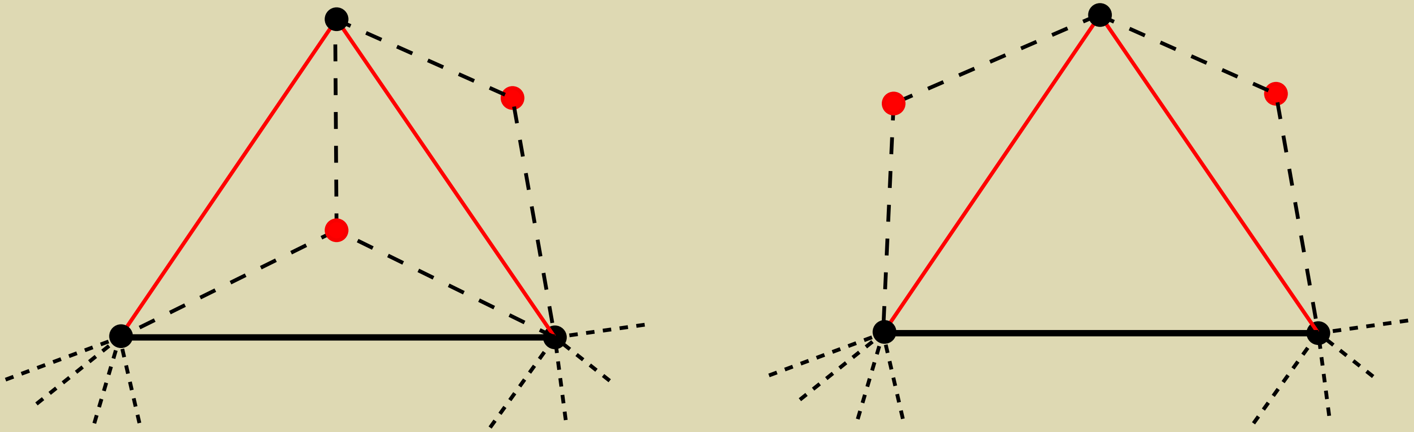
- If degree of x in G_i is two then $|S| \leq 2$.



Case I

- If degree of x in G_i is two then $|S| \leq 2$.

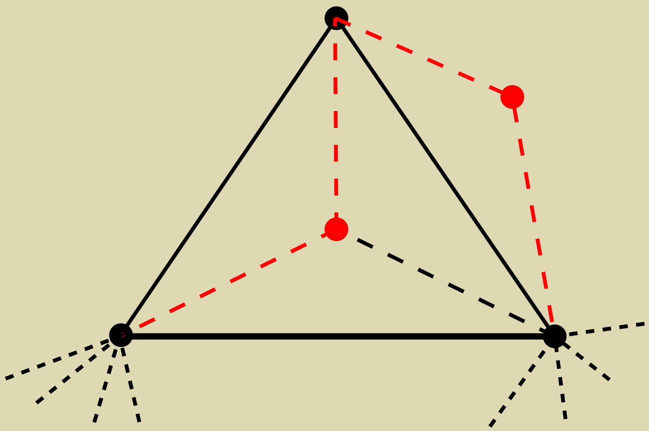
Case I A

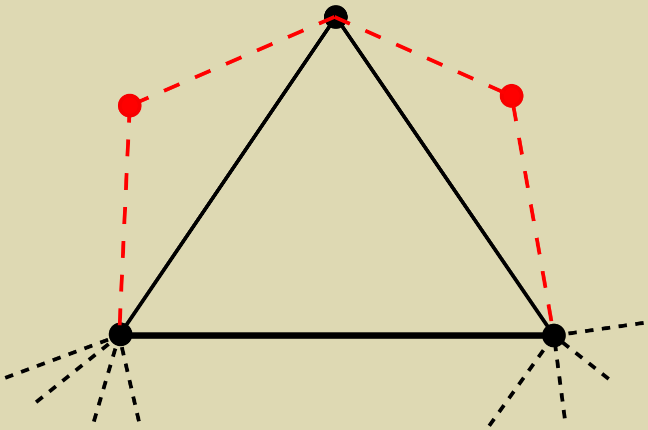


Case I

- If degree of x in G_i is two then $|S| \leq 2$.

Case I A

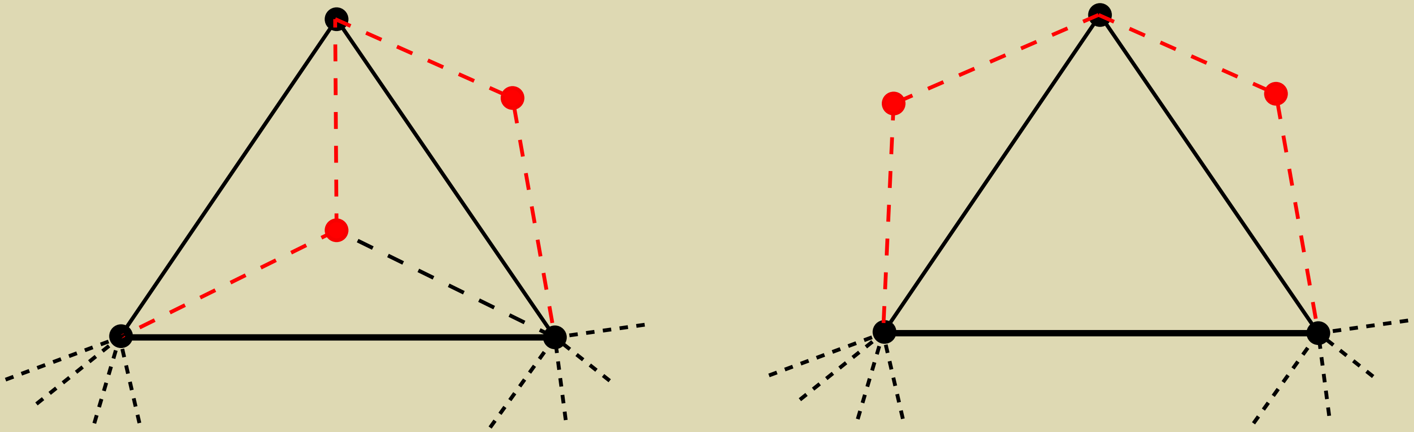




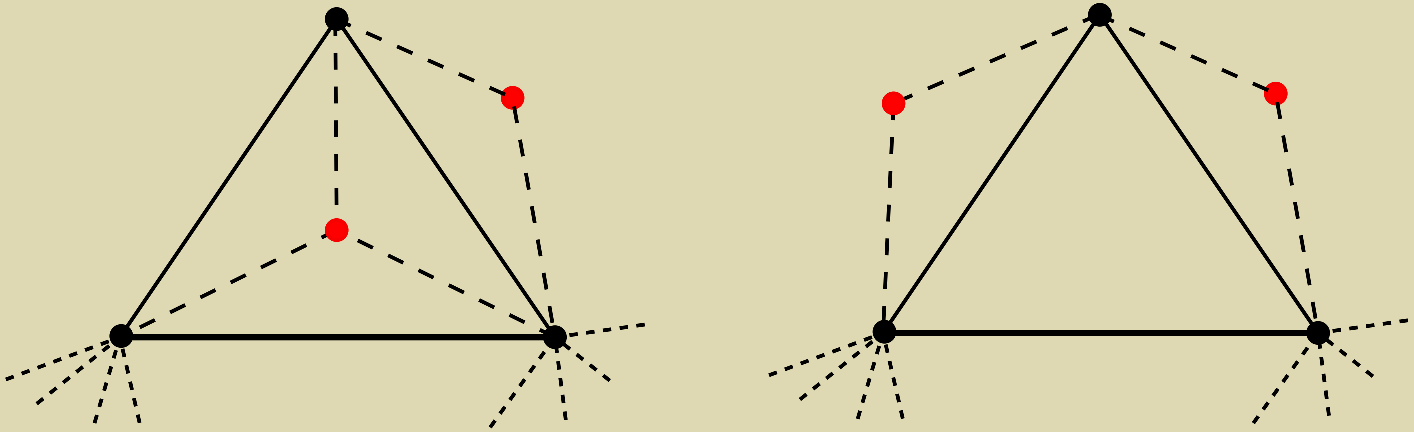
Case I

- If degree of x in G_i is two then $|S| \leq 2$.

Case I A



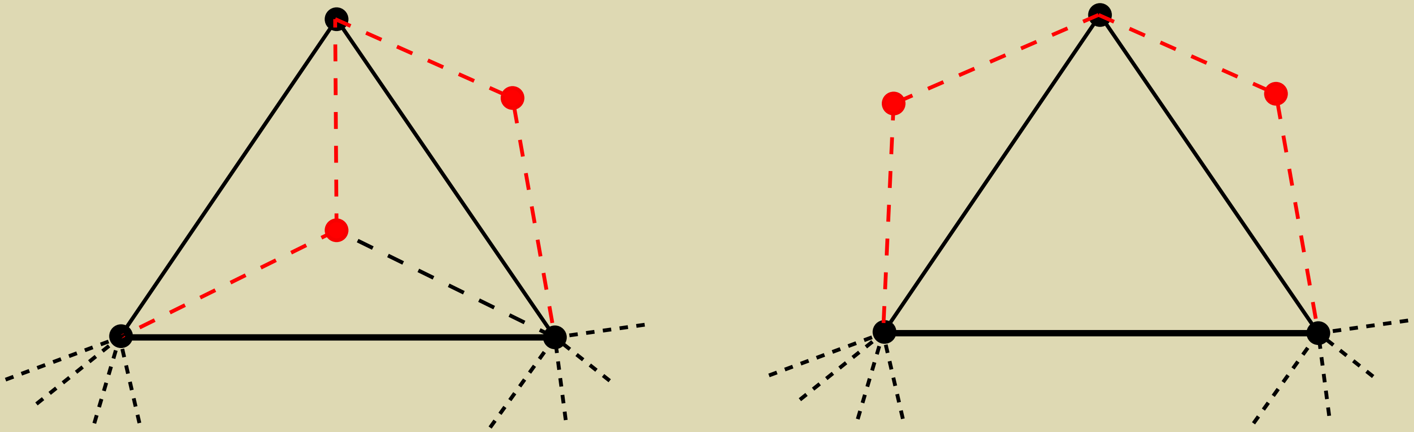
Case IB



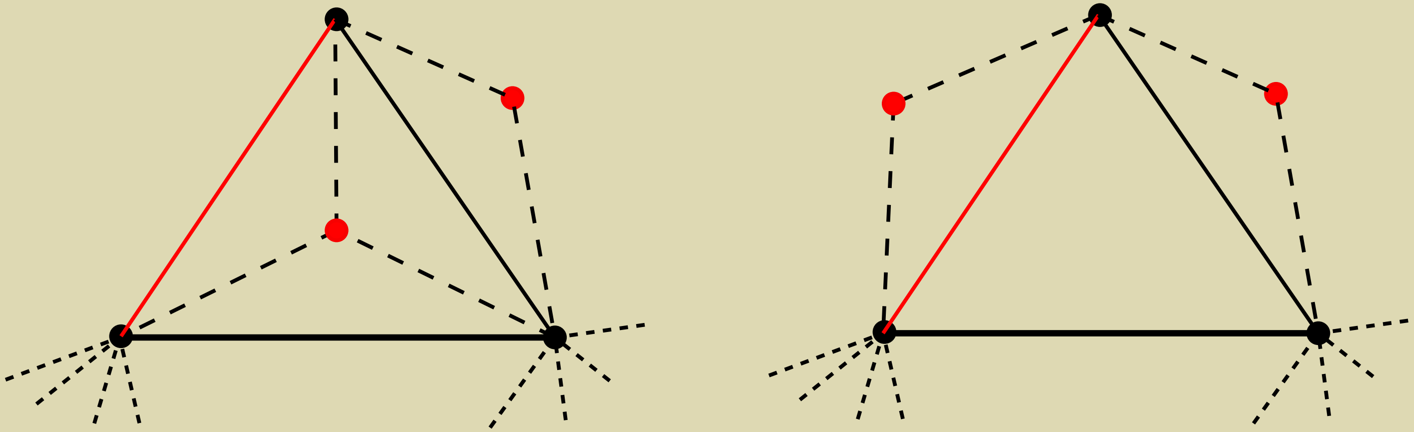
Case I

- If degree of x in G_i is two then $|S| \leq 2$.

Case I A



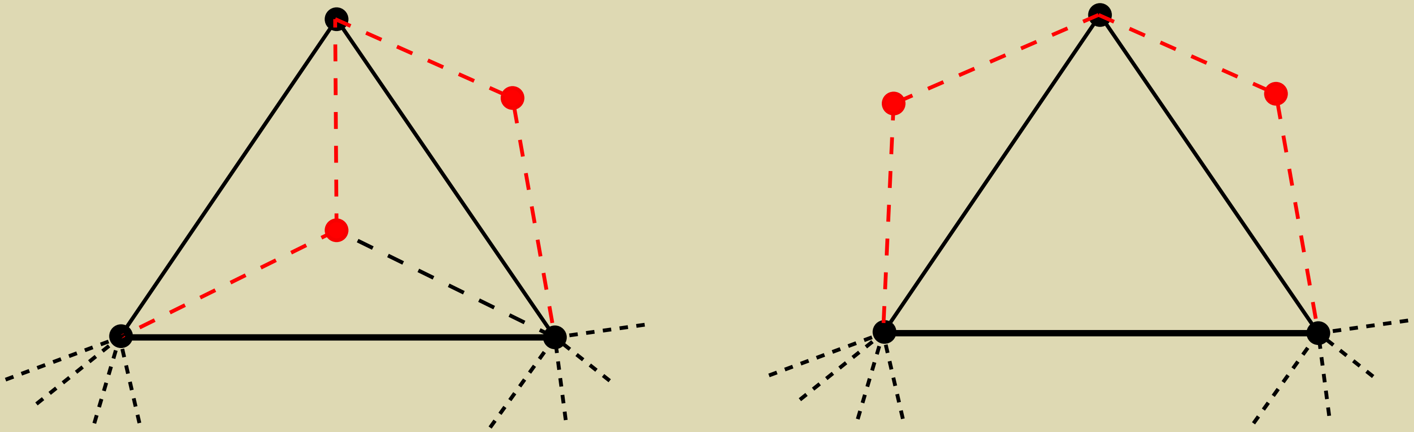
Case IB



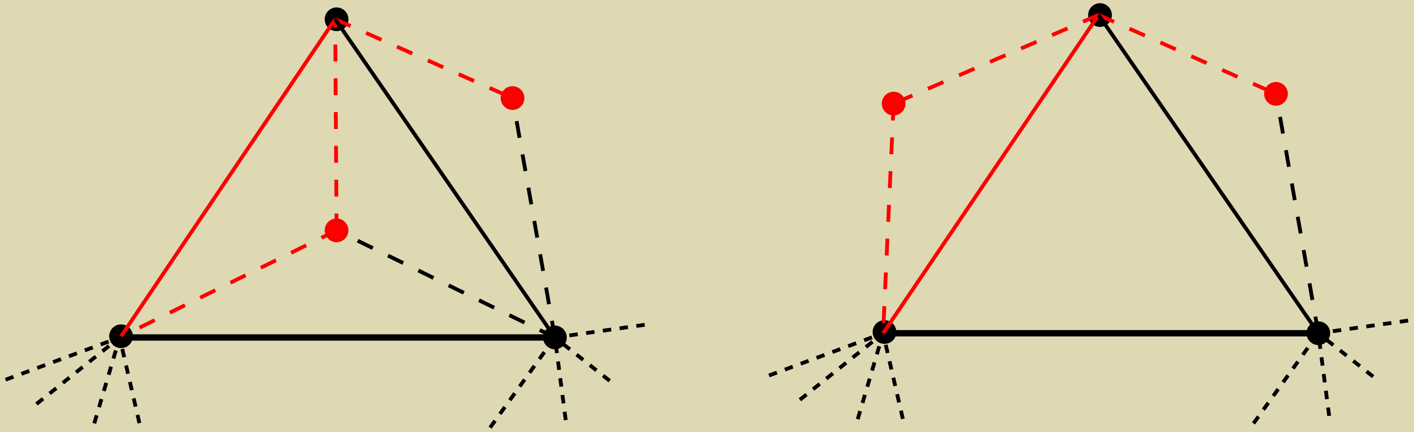
Case I

- If degree of x in G_i is two then $|S| \leq 2$.

Case I A



Case IB



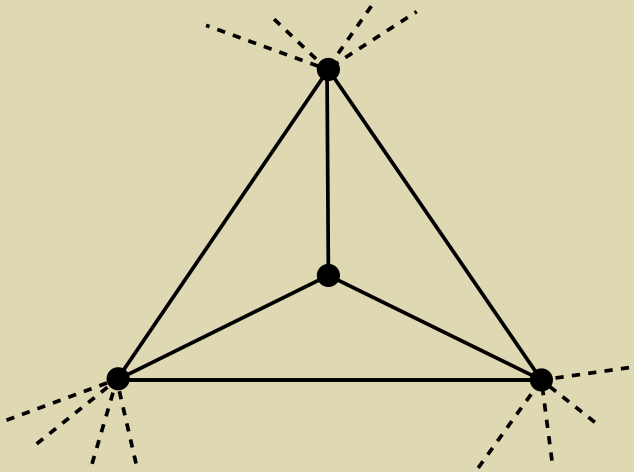
Case II

- If degree of x in G_i is three then $|S| \leq 4$.

Case II

- If degree of x in G_i is three then $|S| \leq 4$.

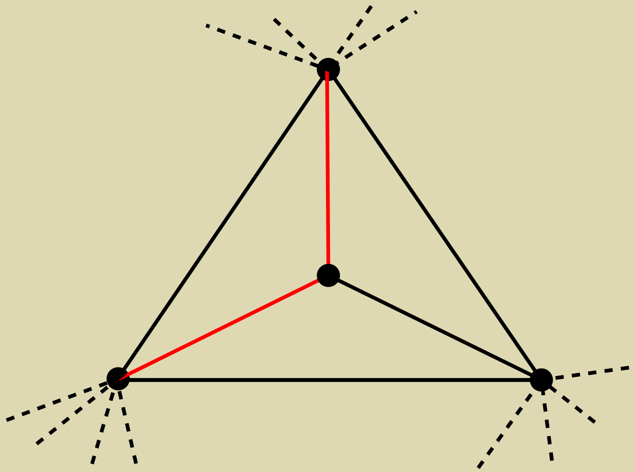
  Case II A; $|S| = 3$



Case II

- If degree of x in G_i is three then $|S| \leq 4$.

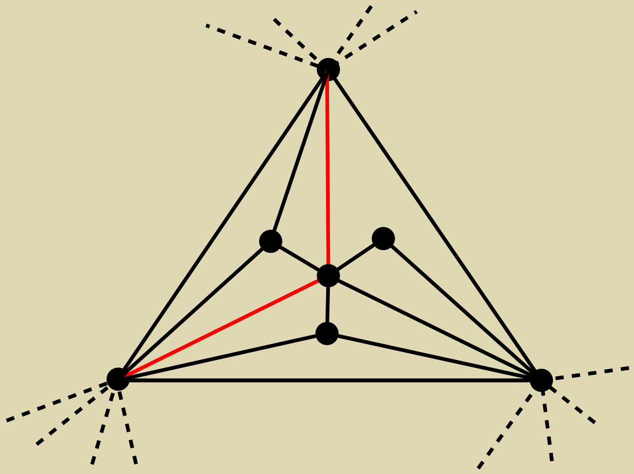
Case II A; $|S| = 3$



Case II

- If degree of x in G_i is three then $|S| \leq 4$.

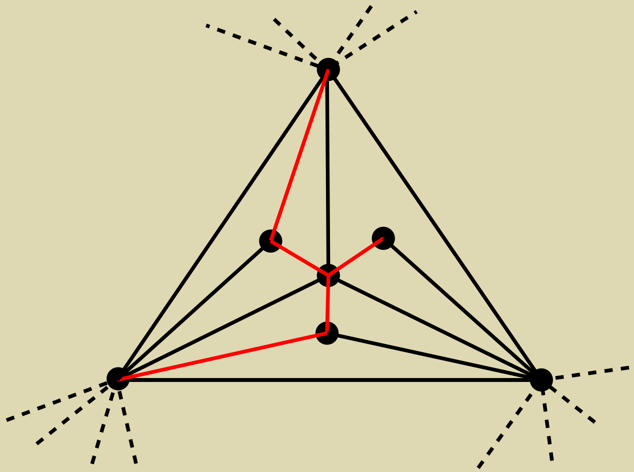
  Case II A; $|S| = 3$



Case II

- If degree of x in G_i is three then $|S| \leq 4$.

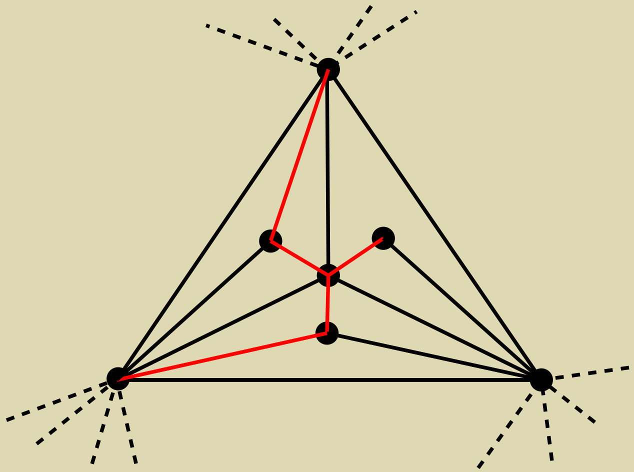
  Case II A; $|S| = 3$



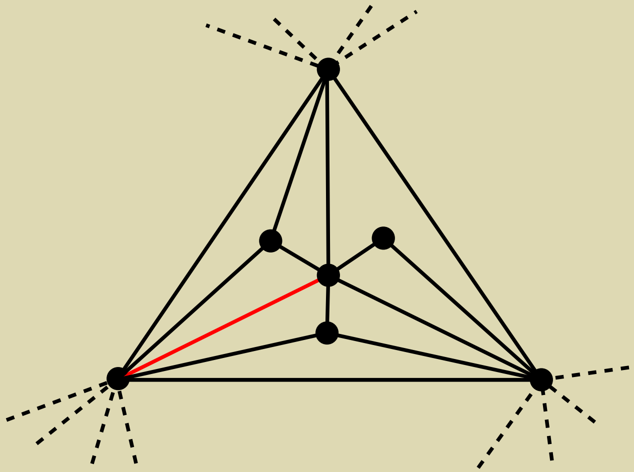
Case II

- If degree of x in G_i is three then $|S| \leq 4$.

  Case II A; $|S| = 3$



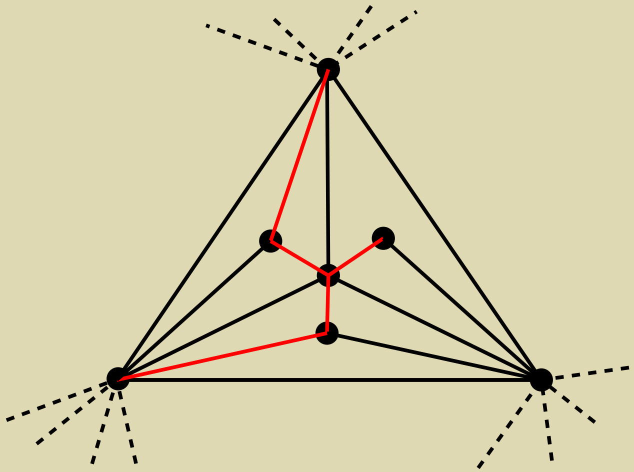
  Case IB; $|S| = 3$



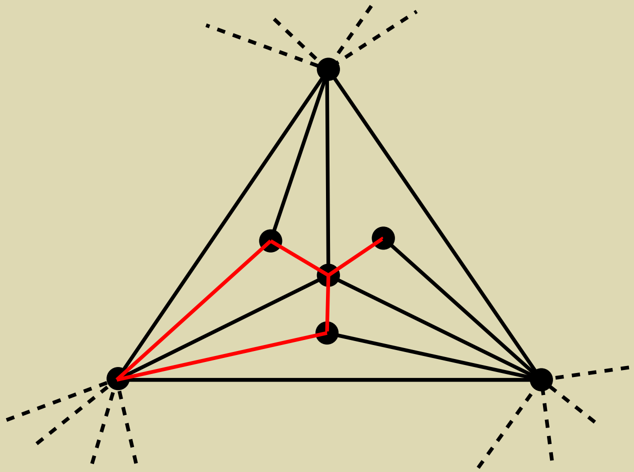
Case II

- If degree of x in G_i is three then $|S| \leq 4$.

Case II A; $|S| = 3$

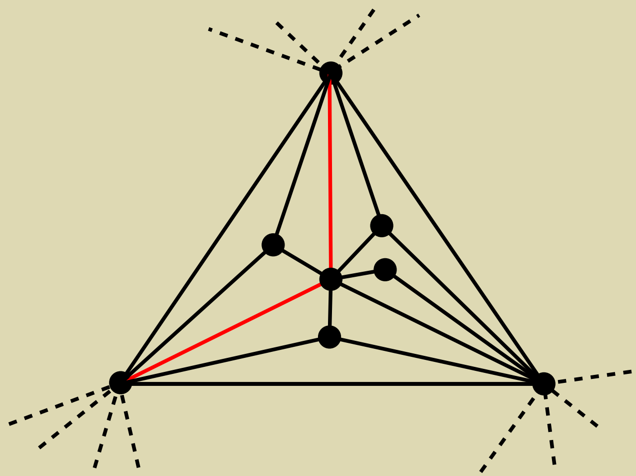


Case IB; $|S| = 3$



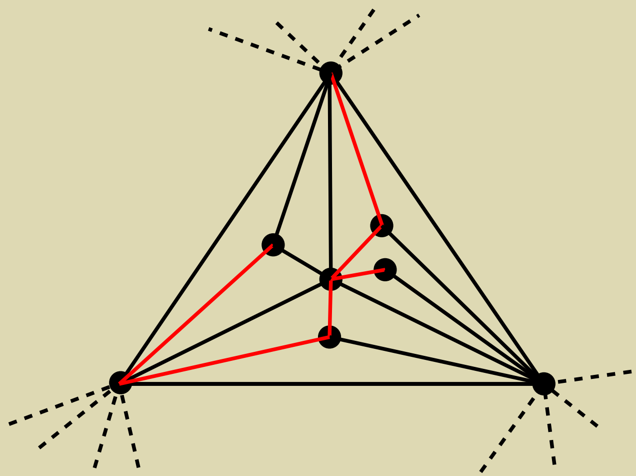
Case II

Case II A; $|S| = 4$

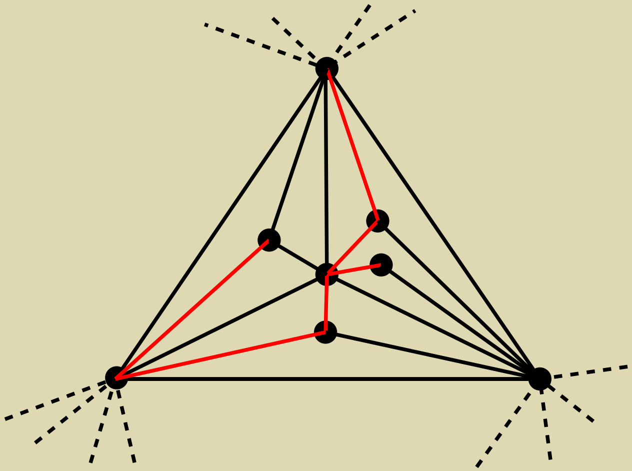


Case II

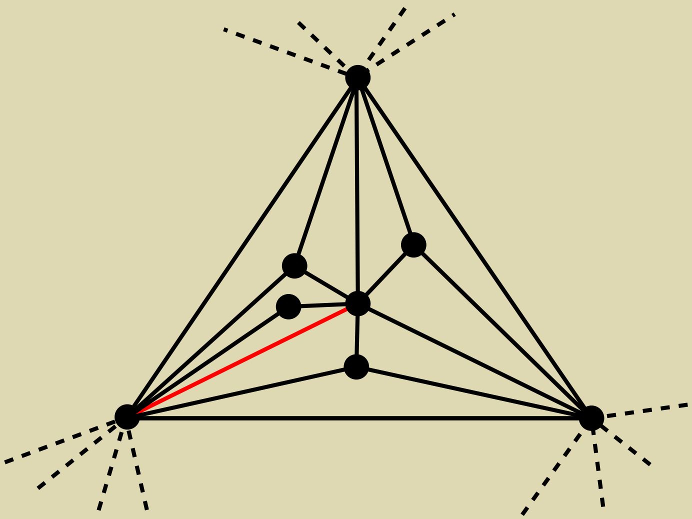
Case II A; $|S| = 4$

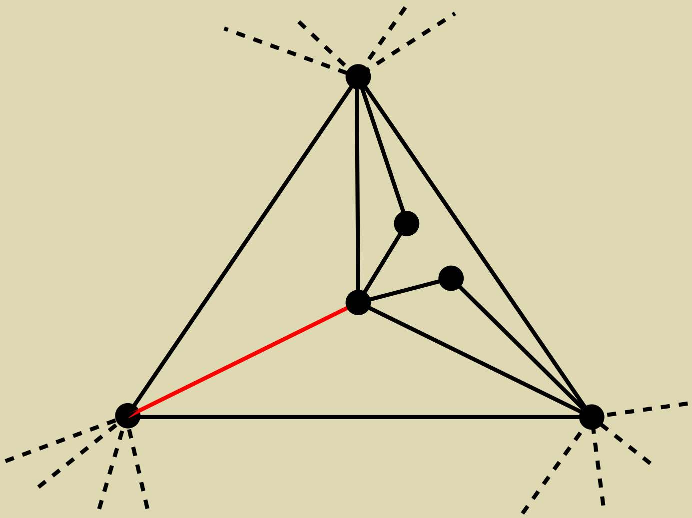


Case II A; $|S| = 4$



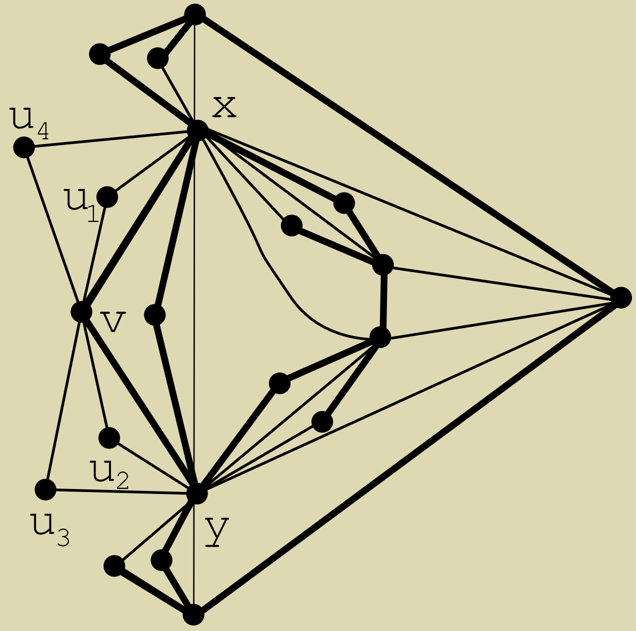
Bad cases





Lower bound

Theorem : There exists an infinite class of 2-connected chordal planar graphs with toughness $t(G) = \frac{1}{2}$ without a 2-walk.



Open problems

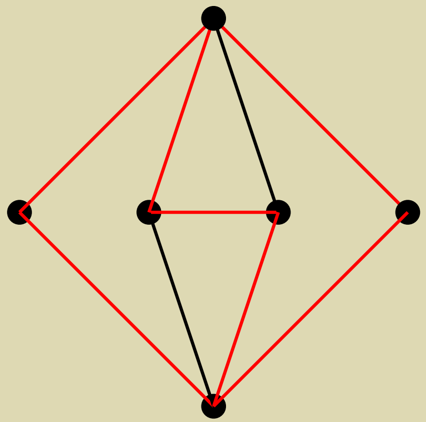
Conjectures:

- There exists a finite constant t_0 such that every t_0-tough graph is hamiltonian.
- Every 2-tough chordal graph is hamiltonian.

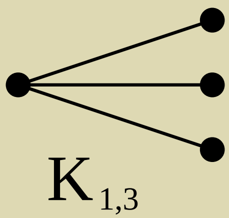
- Every $\frac{1}{k-1}$-tough graph has a k-walk.
- Every 2-tough graph has a 2-walk.
- Every 1-tough chordal graph has a 2-walk.
- Every more then $\frac{1}{2}$-tough chordal planar graph has a 2-walk.

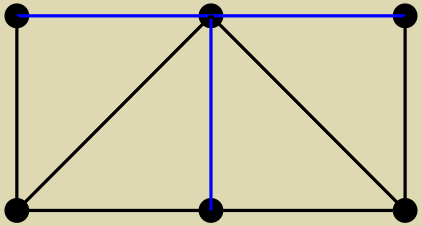
Trestles

- For any integer $r > 1$, an r*-trestle* is a 2-connected graph F with maximum degree $\Delta(F) \leq r$.
- We say that a graph G has an r-trestle if G contains a spanning subgraph which is an r-trestle.



- A graph G is called $K_{1,r}$*-free* if G has no $K_{1,r}$ as an induced subgraph.





Trestles

Ryjáček and Tkáč (2004) proved that

- every 2-connected $K_{1,3}$-free graph has a 3-trestle

They also conjectured that

- every 2-connected $K_{1,r}$-free graph has an r-trestle for every $r \geq 4$.

Trestles

Ryjáček and Tkáč (2004) proved that

- every 2-connected $K_{1,3}$-free graph has a 3-trestle

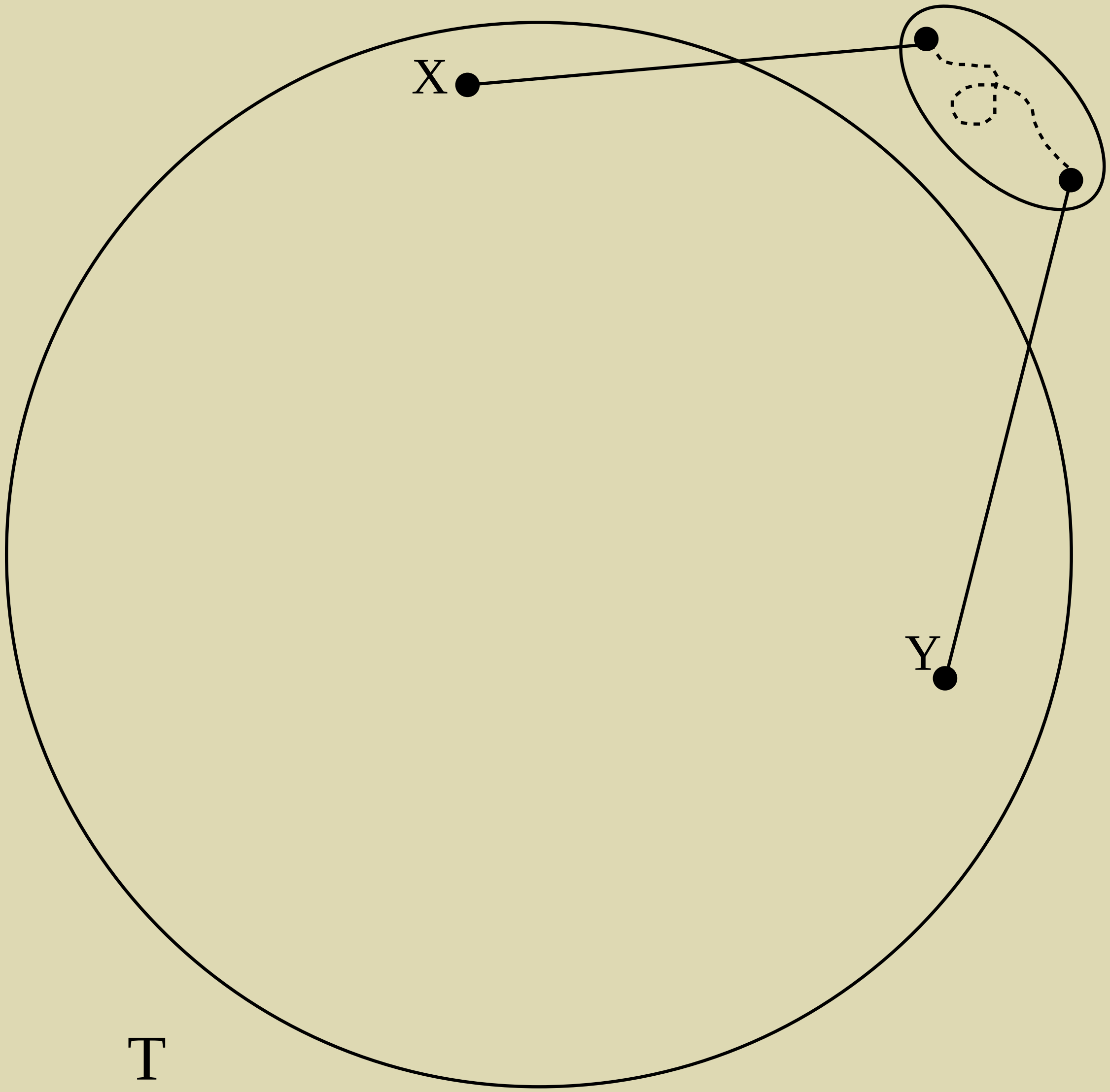
They also conjectured that

- every 2-connected $K_{1,r}$-free graph has an r-trestle for every $r \geq 4$.

Theorem. Every 2-connected $K_{1,r}$-free graph has an r-trestle for every $r \geq 2$.

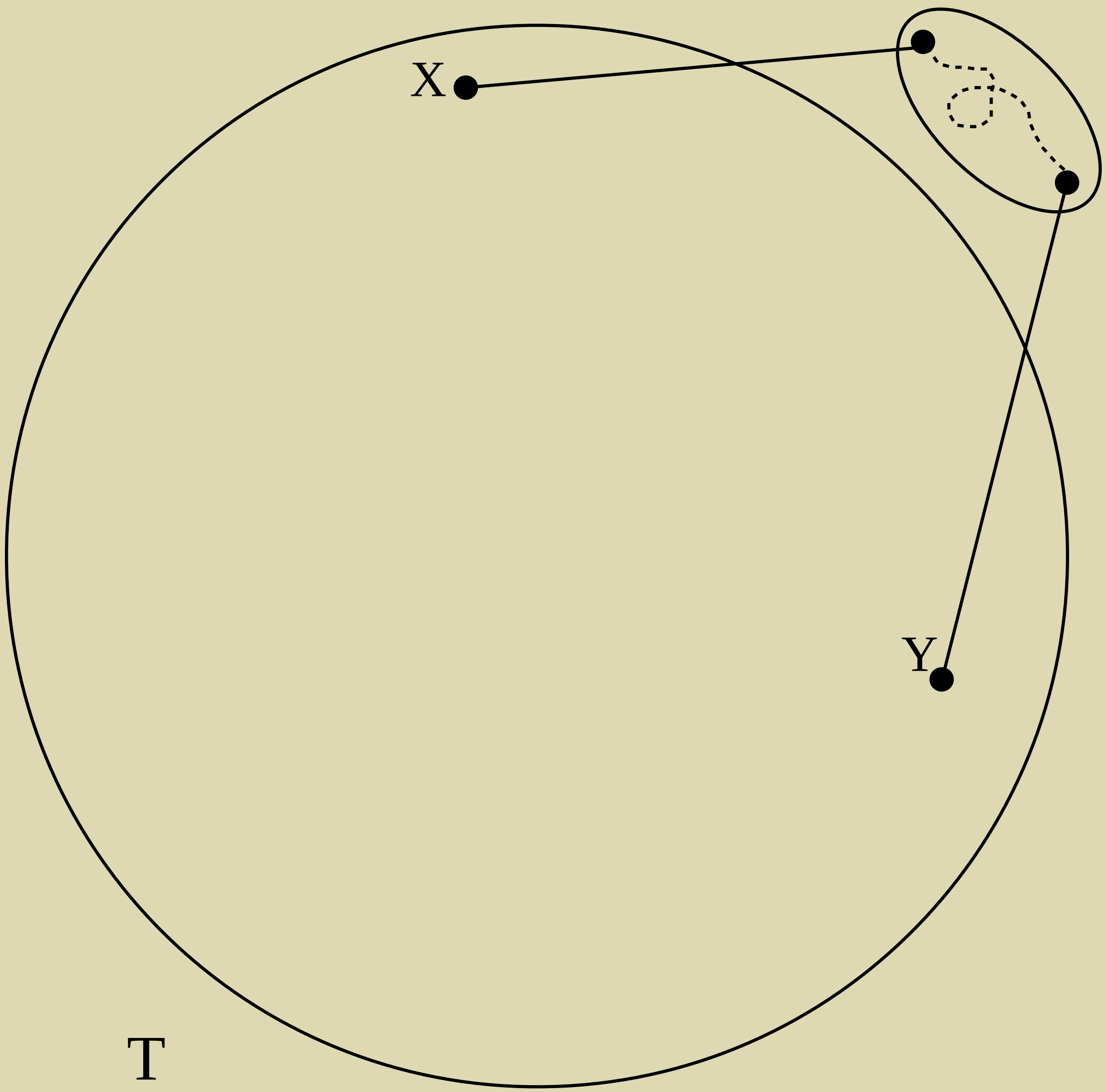
Proof - the main trick = good choice

1. $|V(T)|$ is maximal,



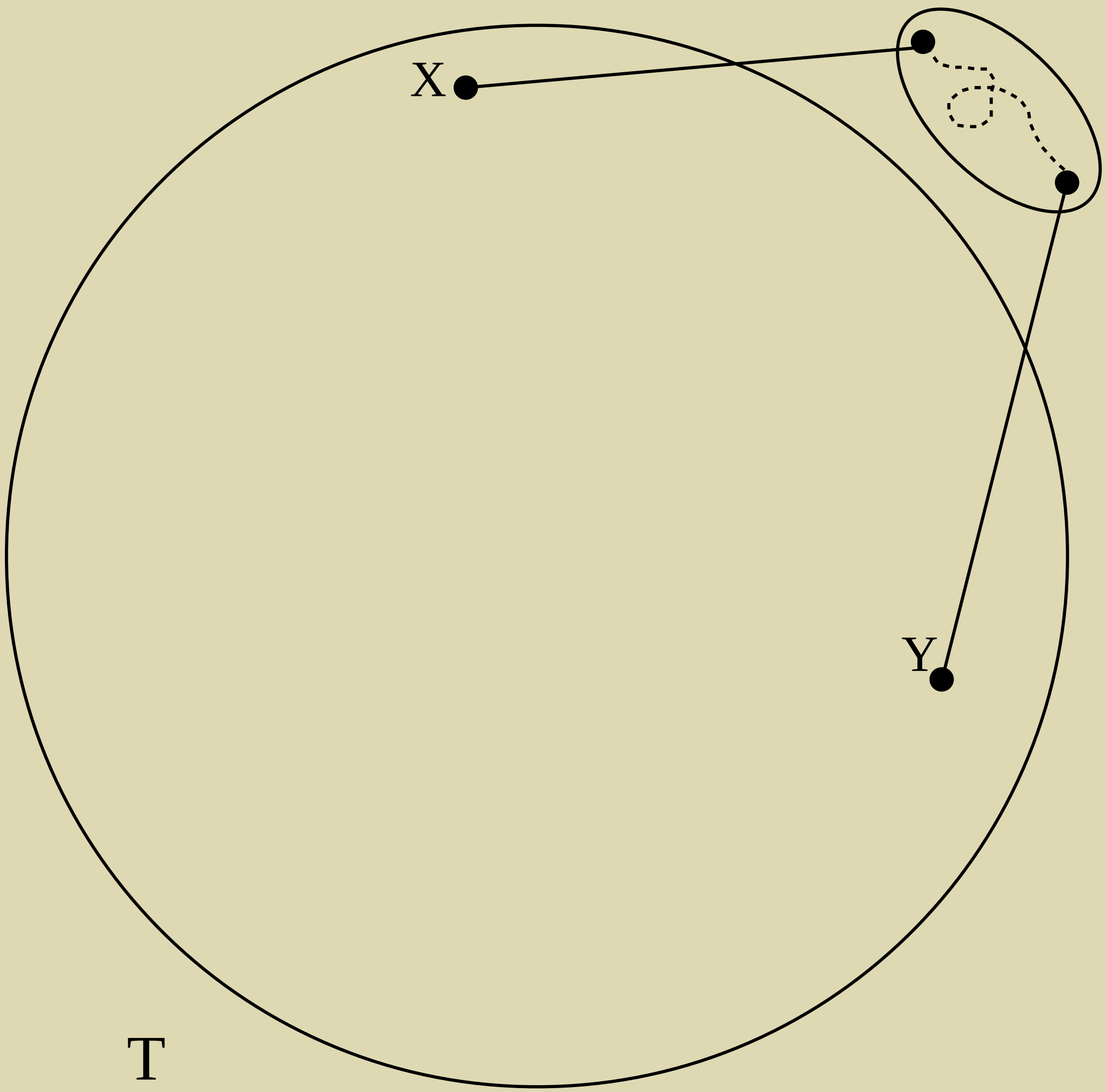
Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.



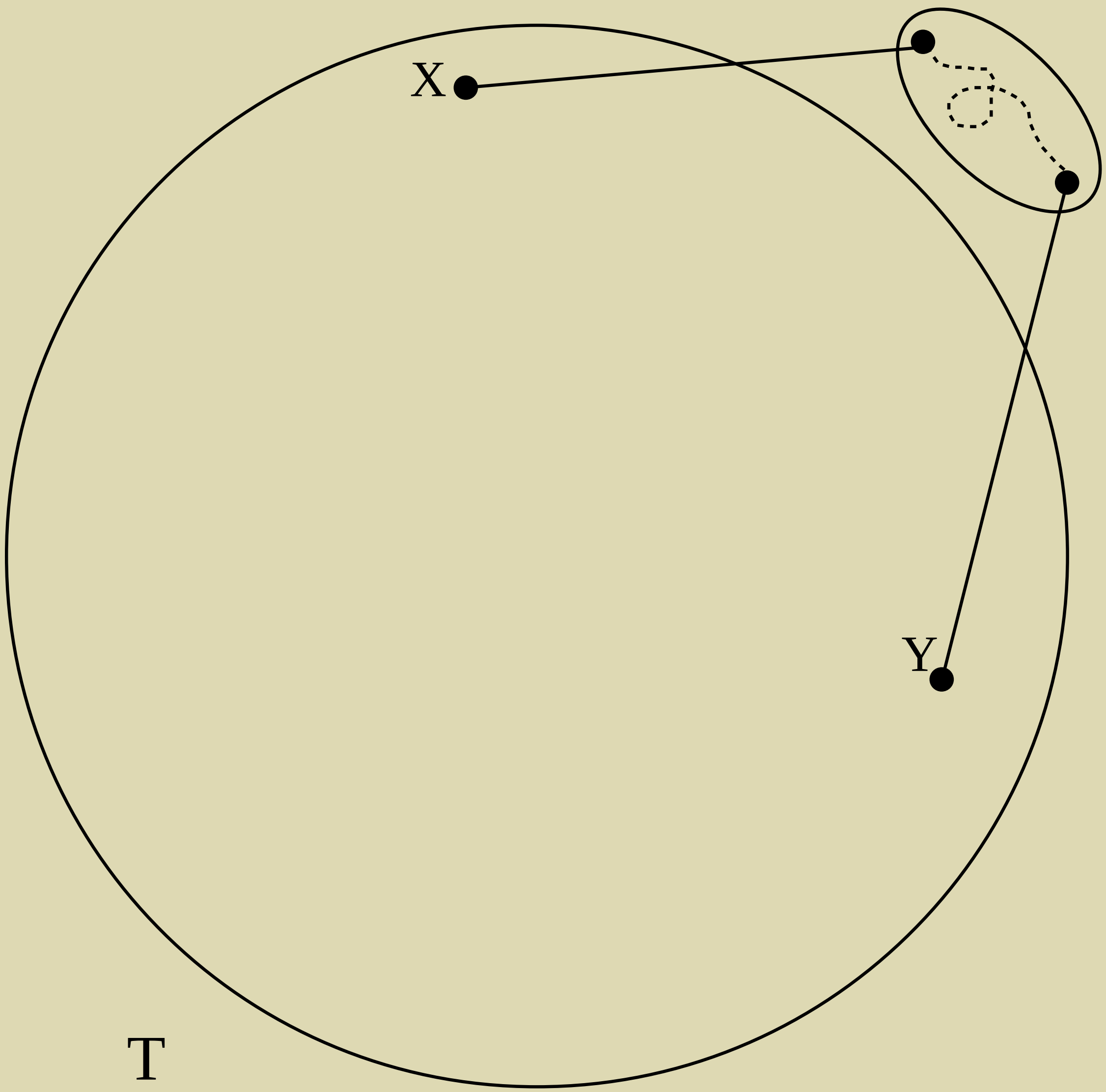
Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal



Proof - the main trick = good choice

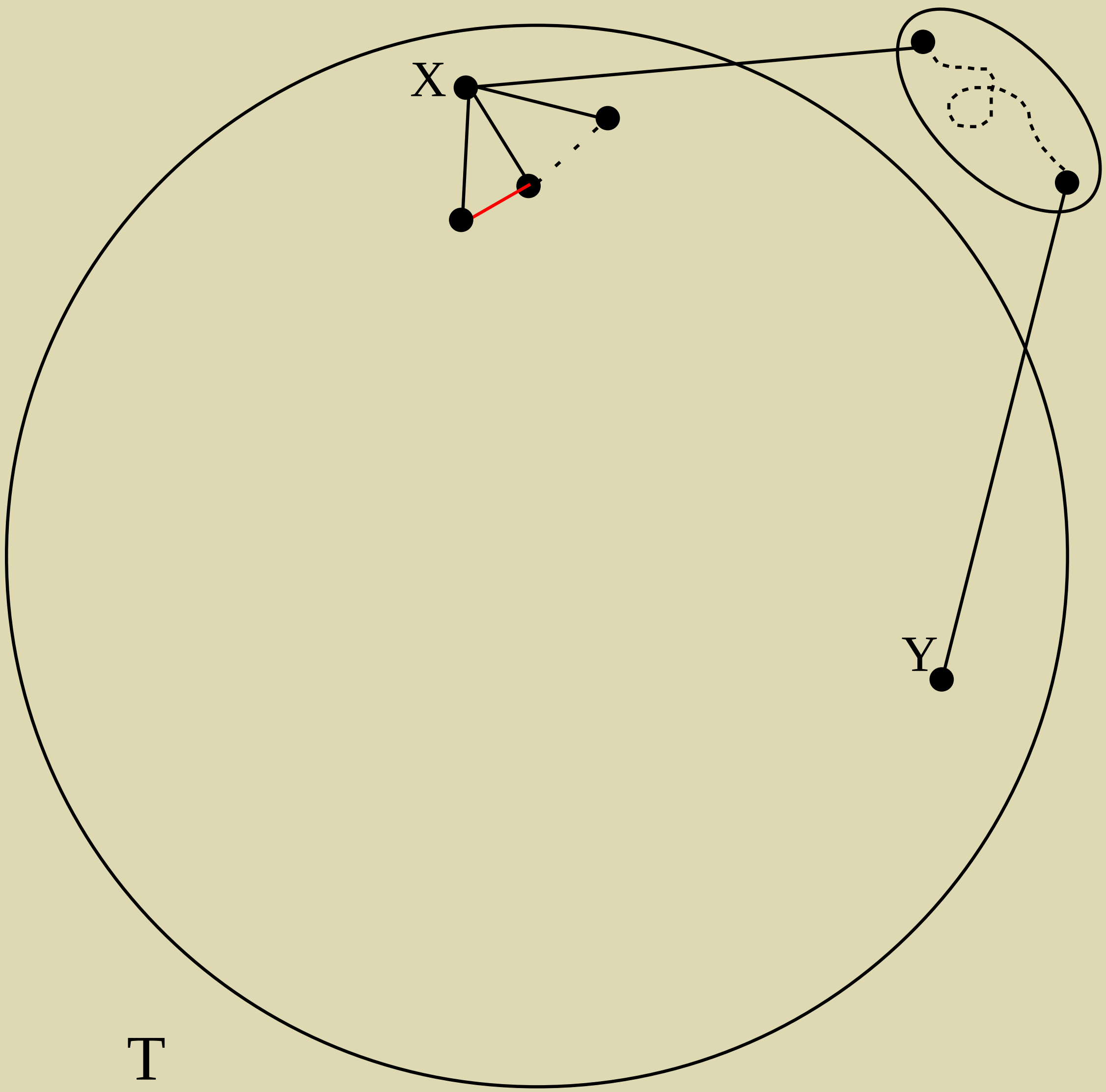
1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.



Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

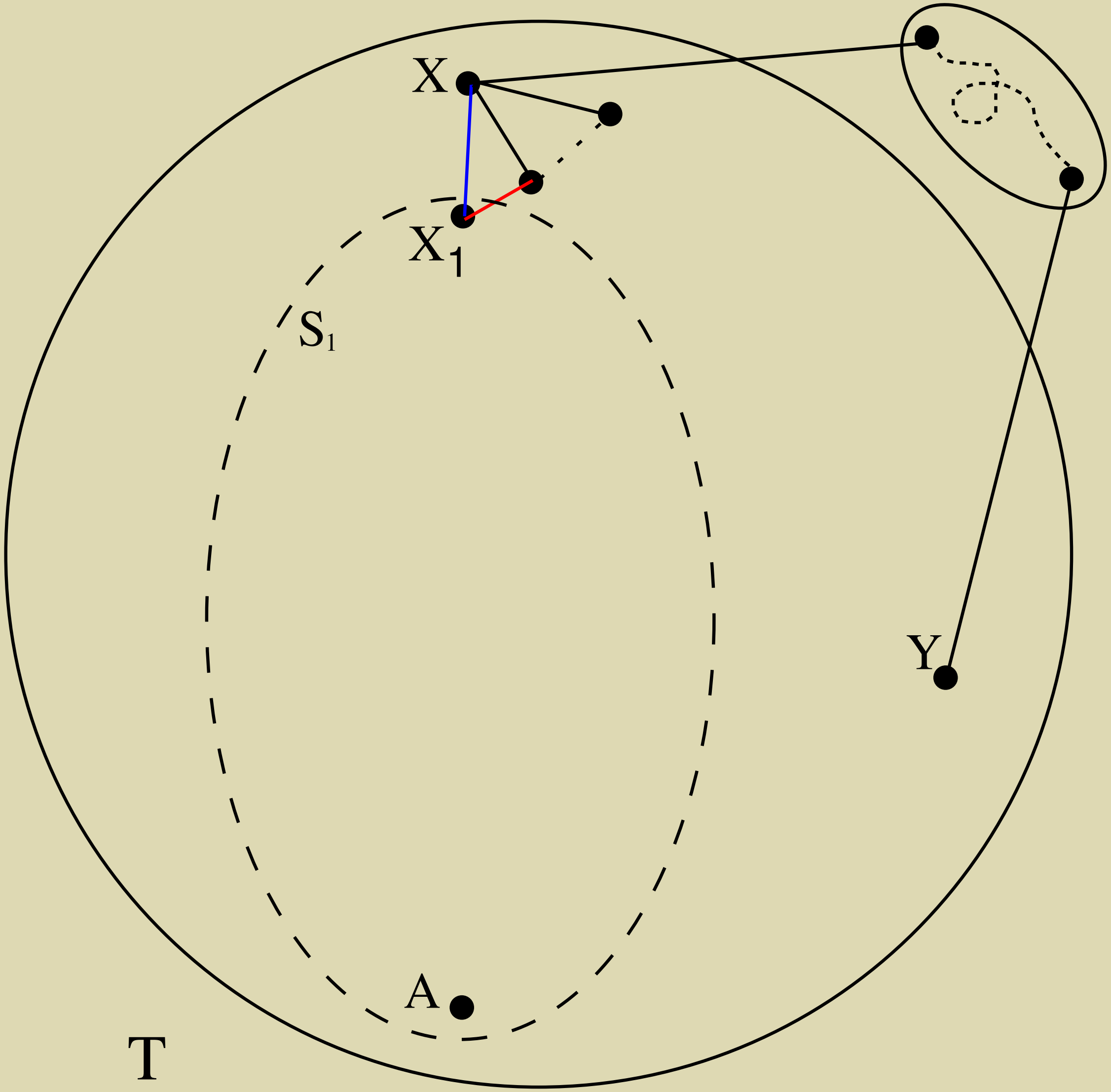
$d_T(X) = r$



Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

$d_T(X) = r$

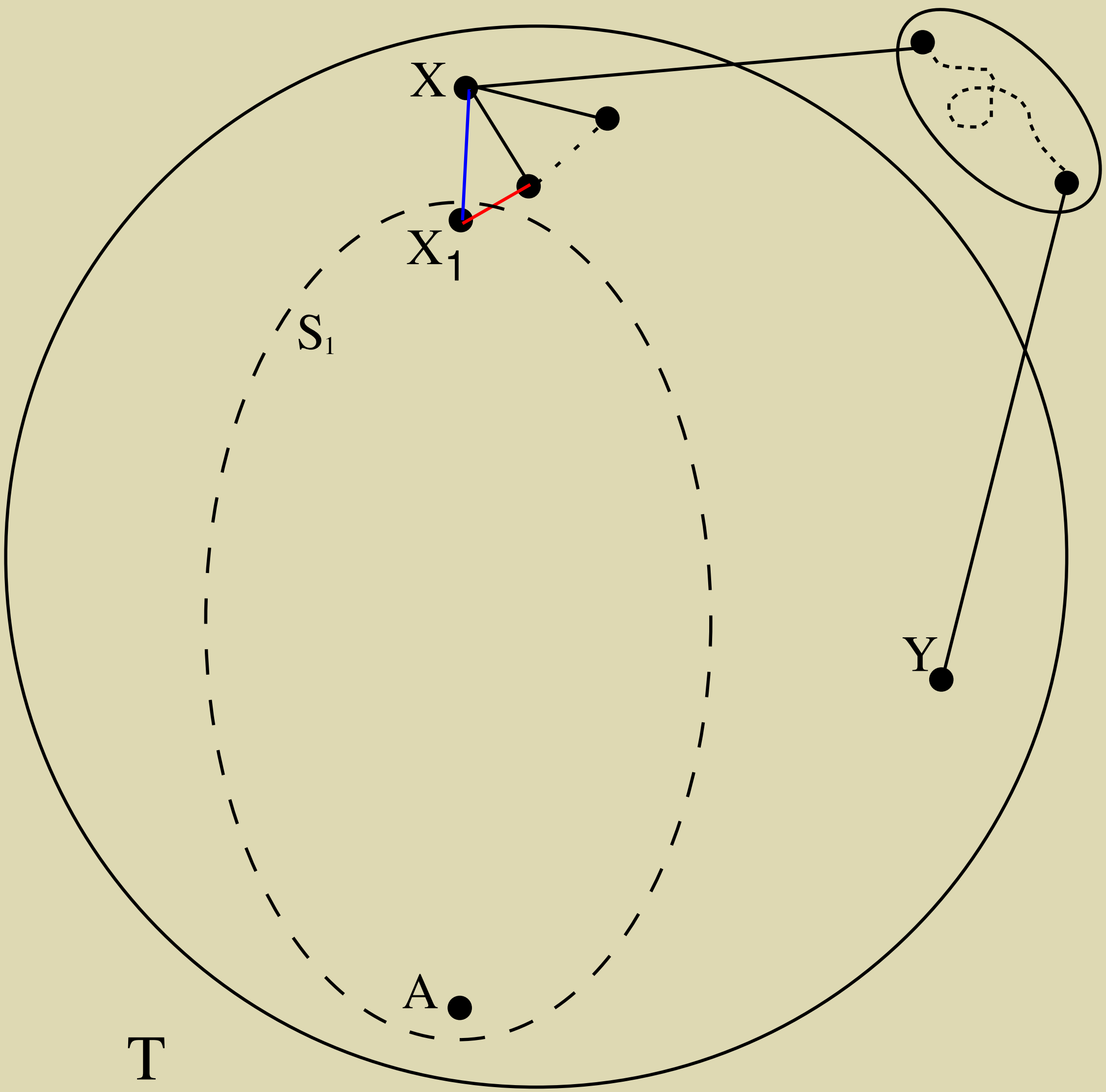


Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

$d_T(X) = r$

Where is the vertex Y ?

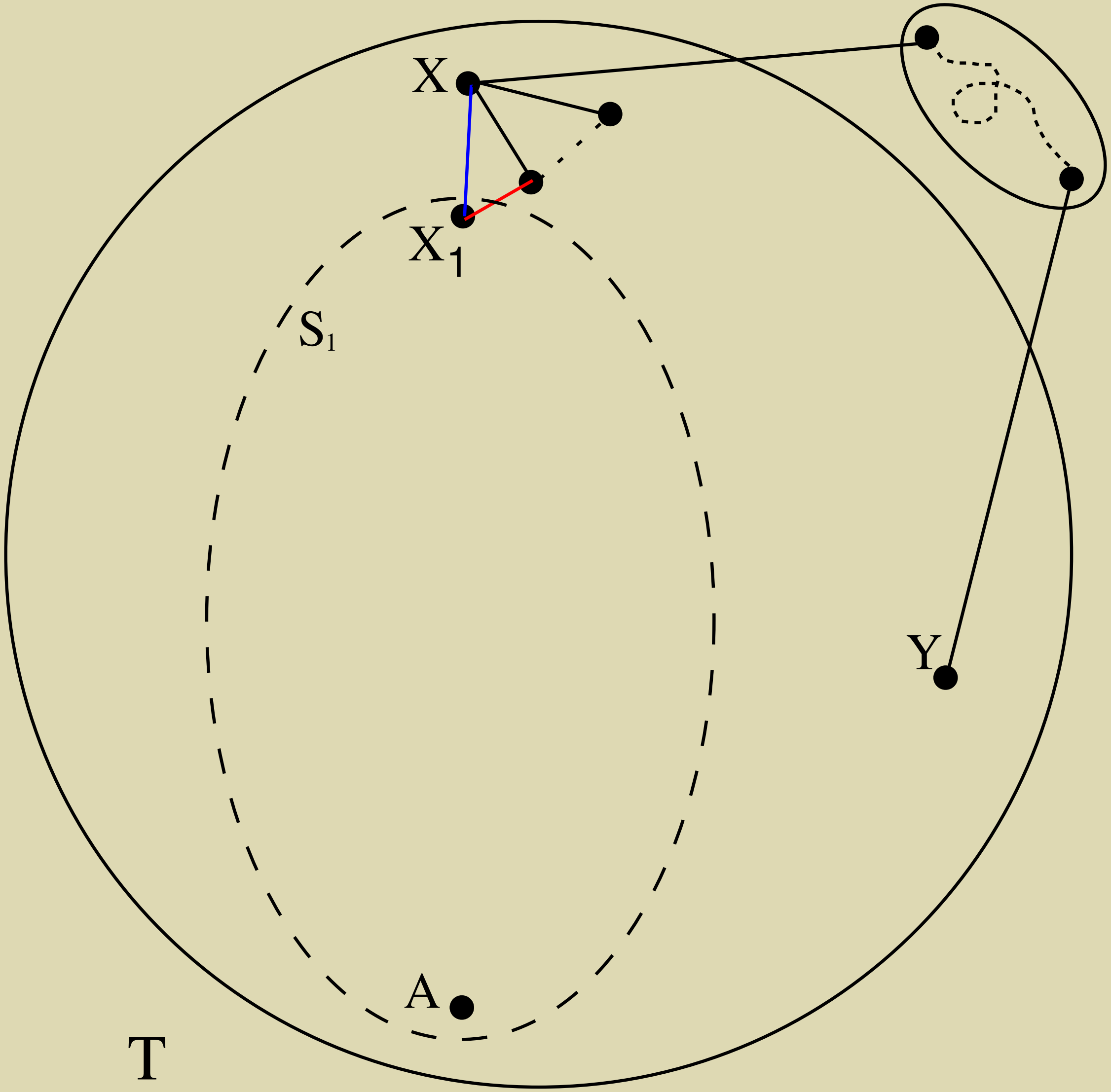


Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

$d_T(X) = r$

$Y \notin S_1$ or $Y = A$



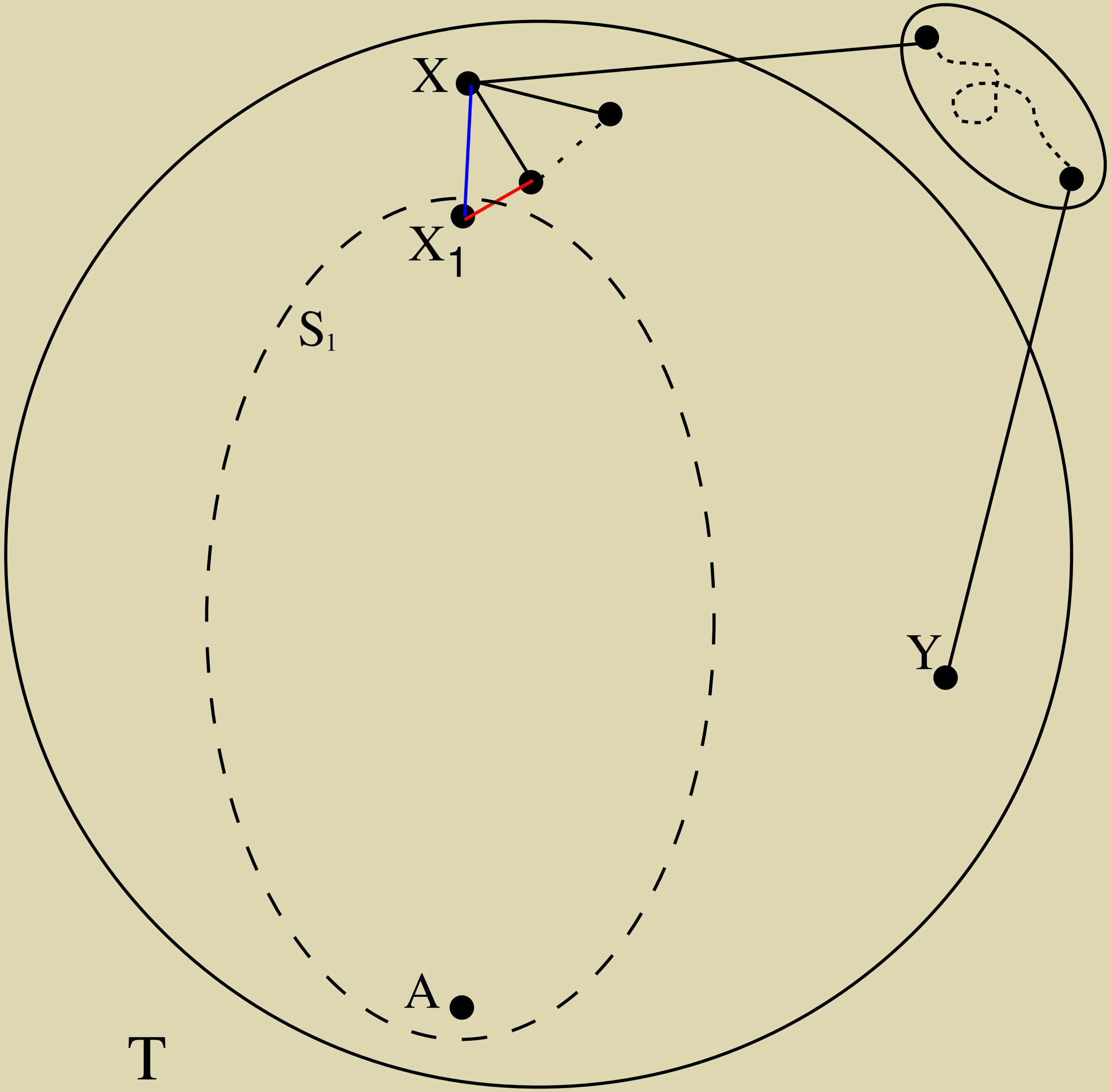
Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

$d_T(X) = r$

$Y \notin S_1$ or $Y = A$

$d_T(X_1) = r$



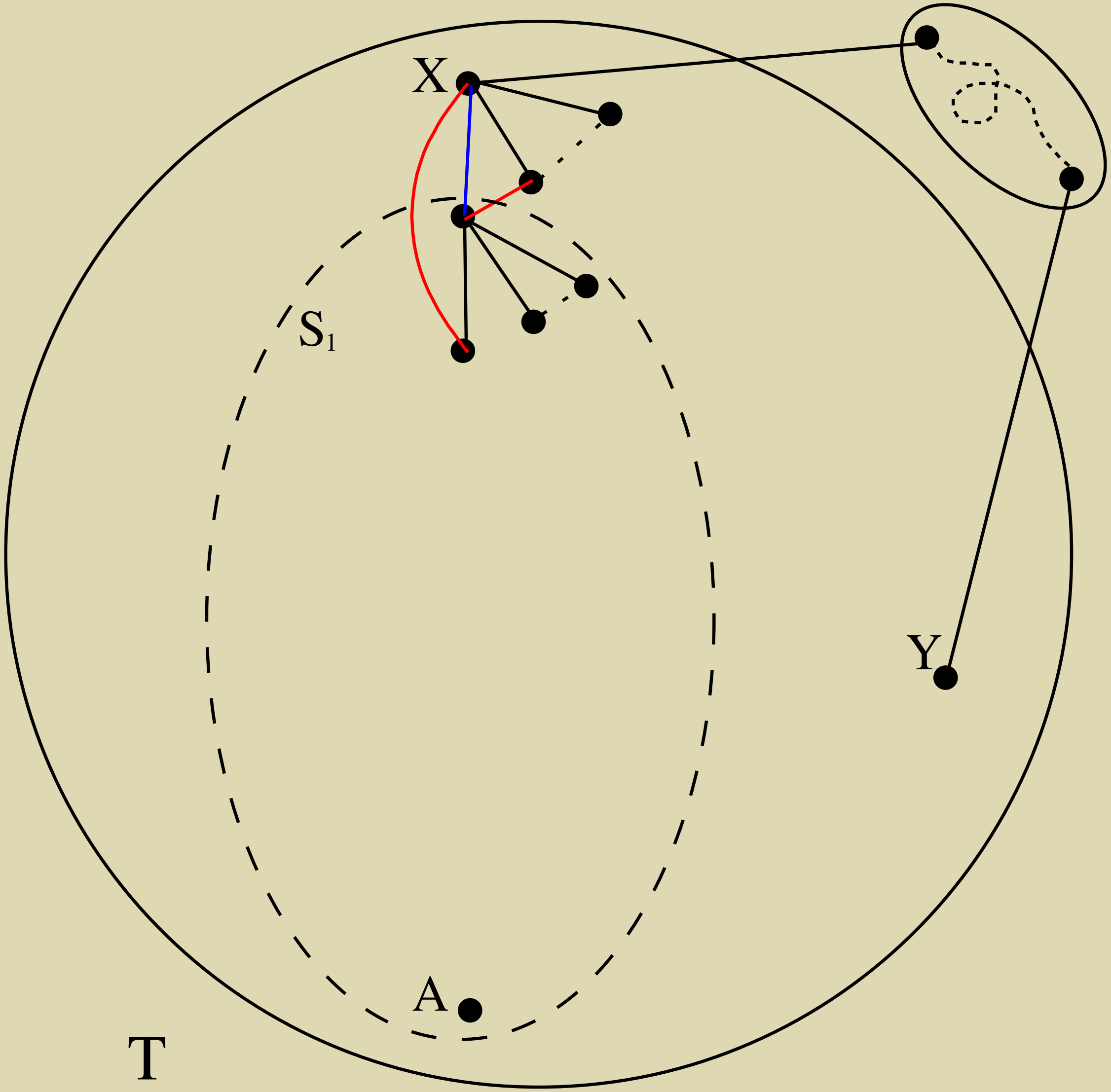
Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

$d_T(X) = r$

$Y \notin S_1$ or $Y = A$

$d_T(X_1) = r$



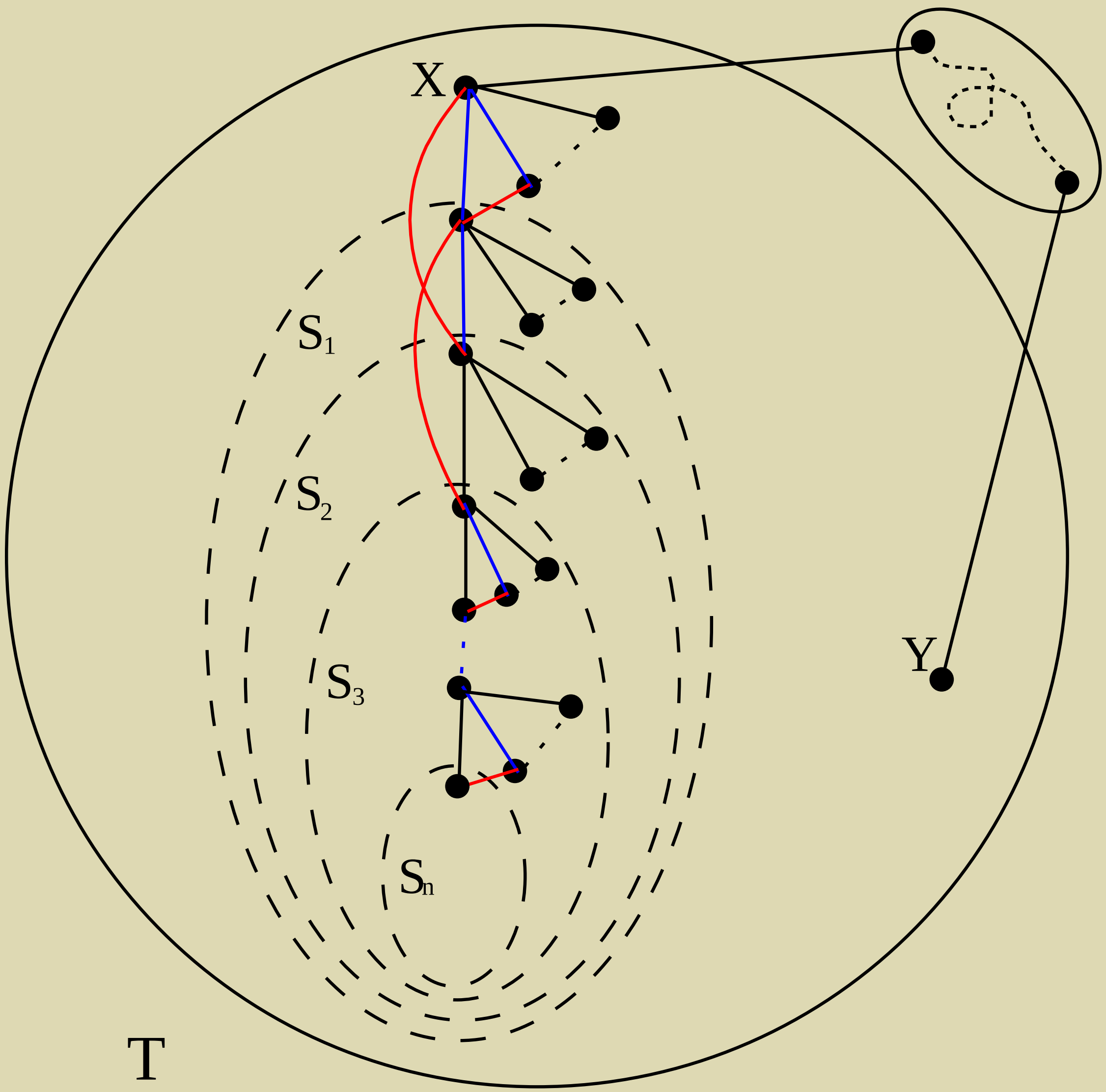
Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

$d_T(X) = r$

$Y \notin S_1$ or $Y = A$

$S_1 \supsetneq S_2 \supsetneq \ldots \supsetneq S_i$.



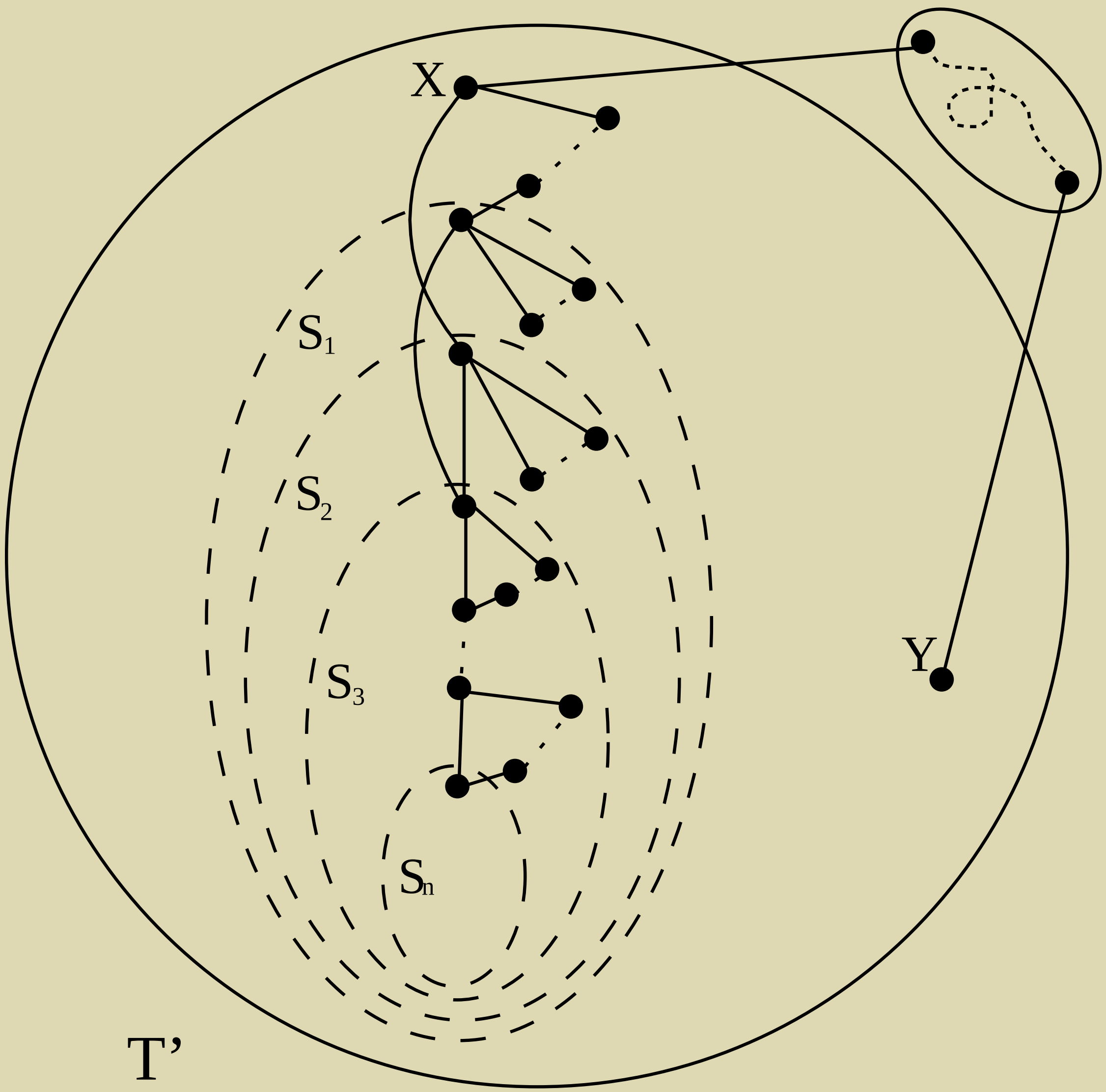
Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

$d_T(X) = r$

$Y \notin S_1$ or $Y = A$

$S_1 \supsetneq S_2 \supsetneq \ldots \supsetneq S_i$.



Proof - the main trick = good choice

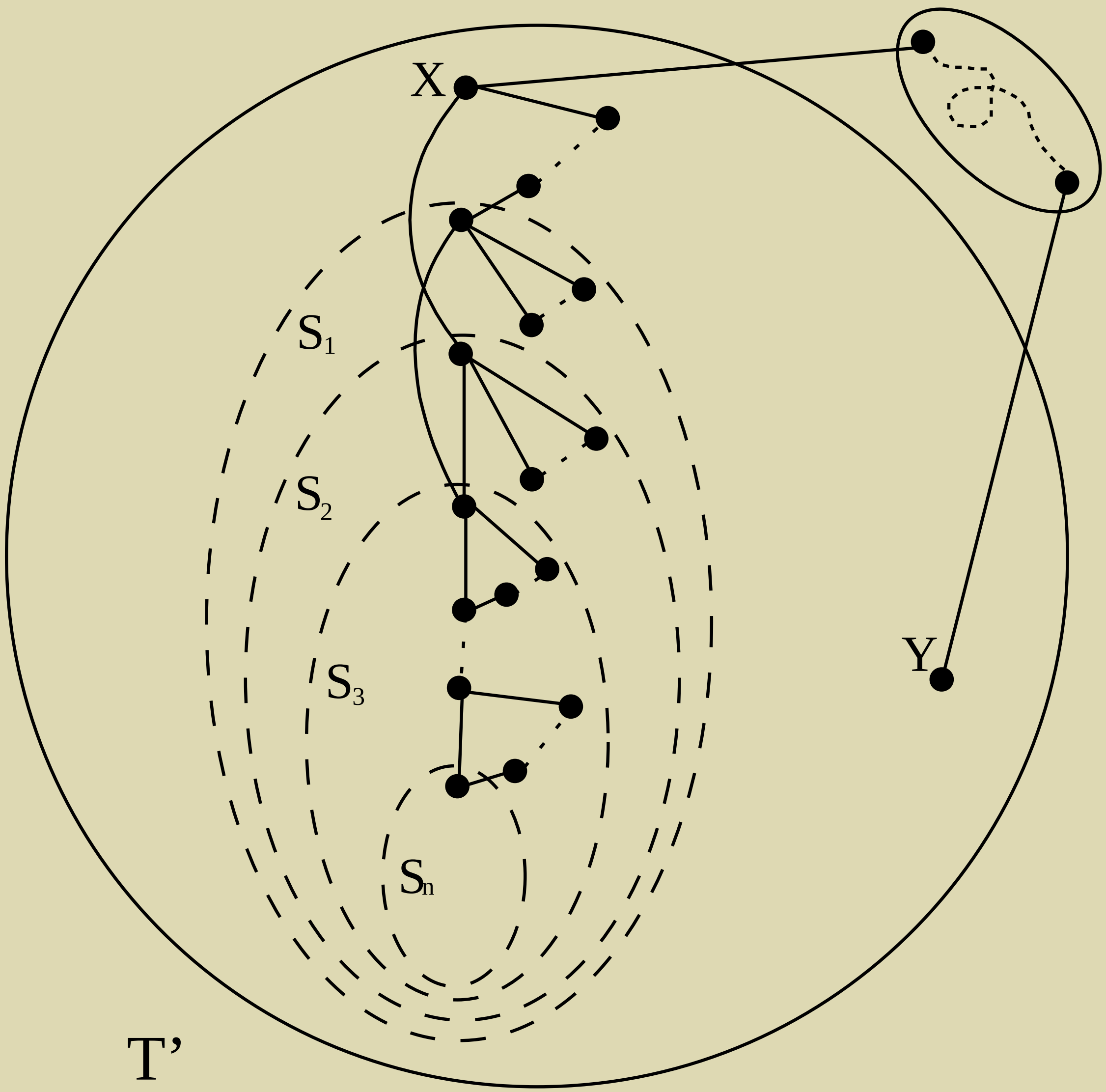
1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

$d_T(X) = r$

$Y \notin S_1$ or $Y = A$

$S_1 \supsetneq S_2 \supsetneq \ldots \supsetneq S_i$.

T' remains 2-connected



Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

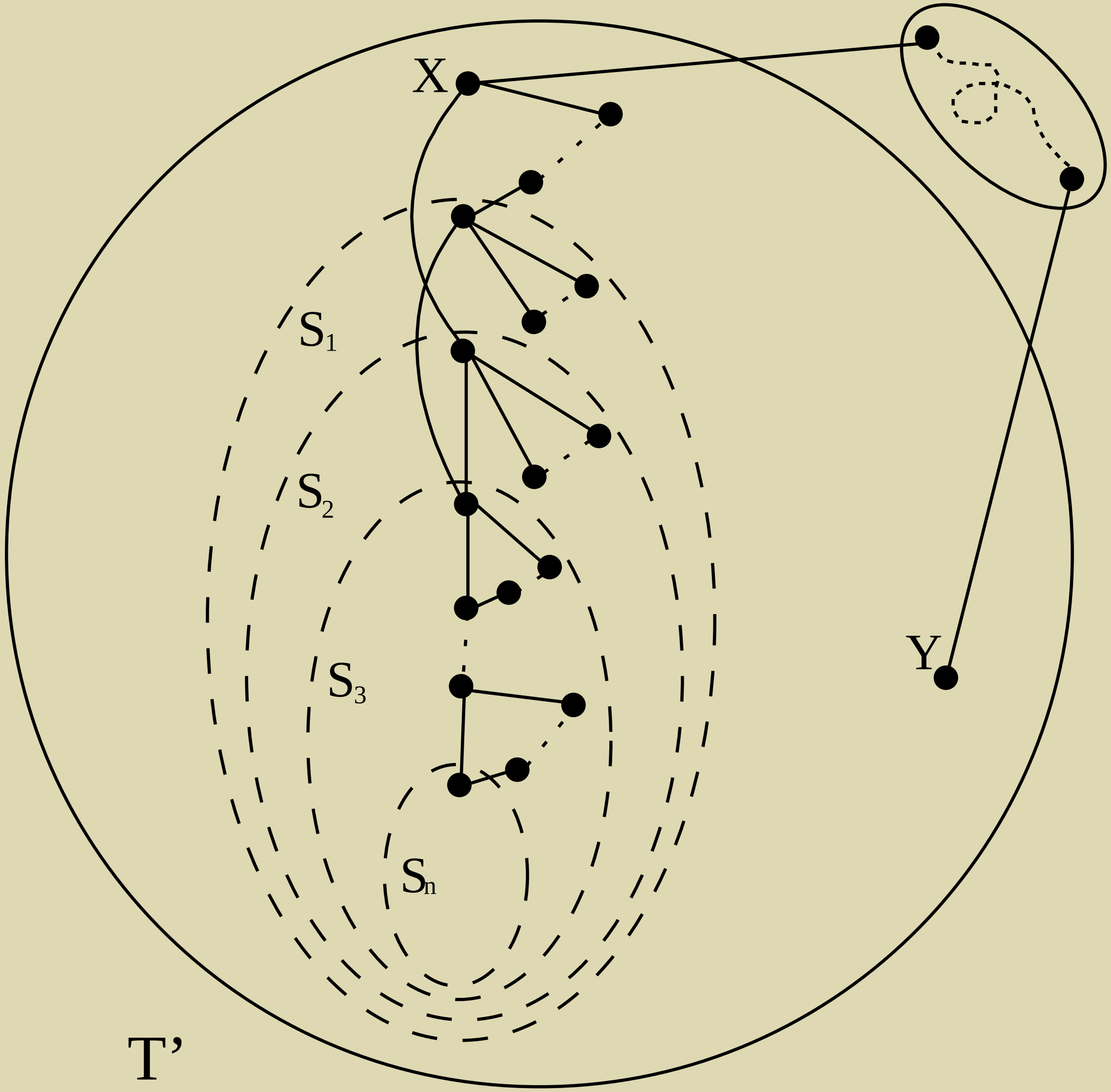
$d_T(X) = r$

$Y \notin S_1$ or $Y = A$

$S_1 \supsetneq S_2 \supsetneq \ldots \supsetneq S_i$.

T' remains 2-connected

$d_{T'}(X) = r - 1$



Proof - the main trick = good choice

1. $|V(T)|$ is maximal,
2. $|E(T)|$ is minimal.
3. $d_T(X) + d_T(Y)$ is minimal
4. $d_T(X) \geq d_T(Y)$.

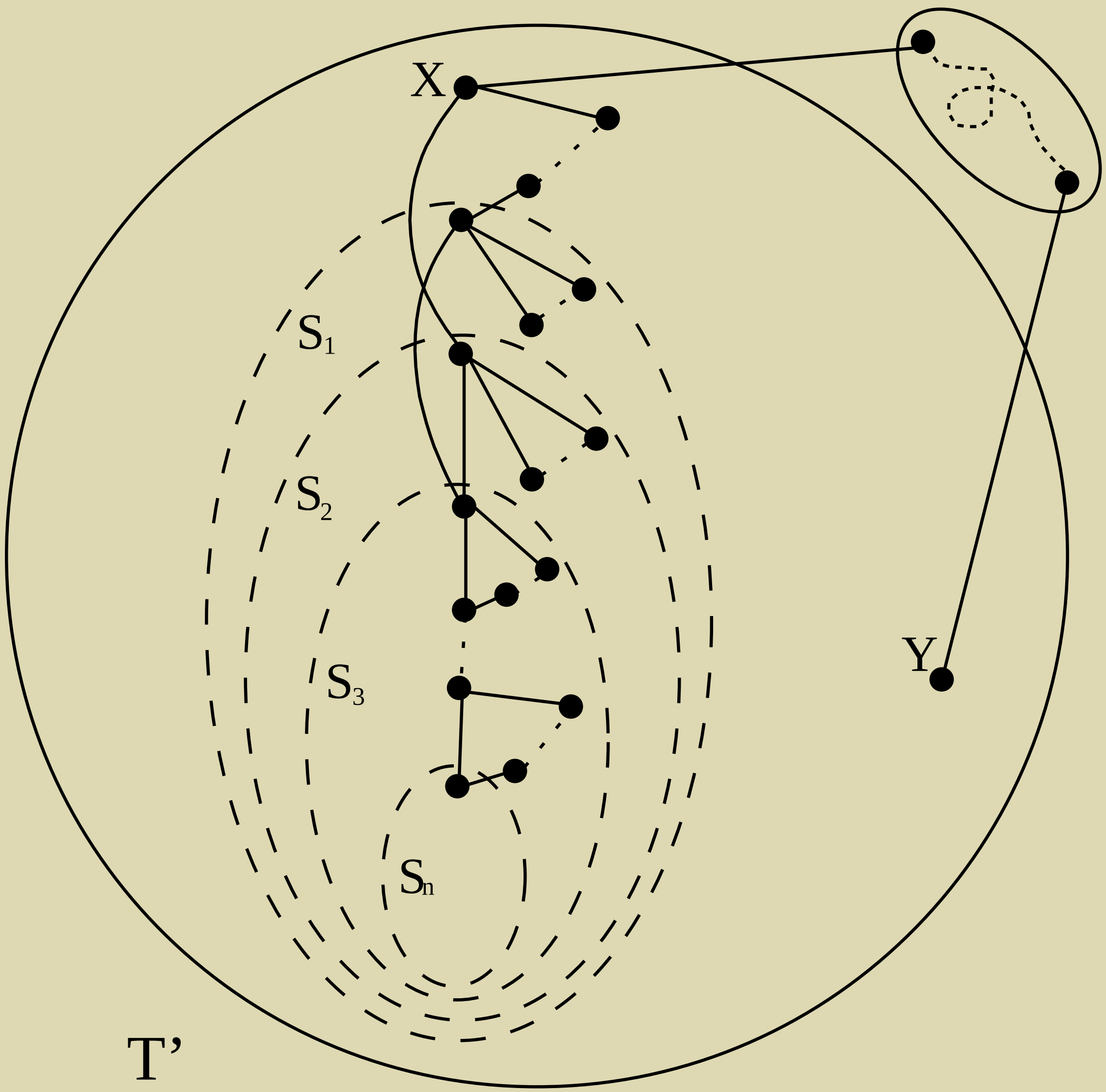
$d_T(X) = r$

$Y \notin S_1$ or $Y = A$

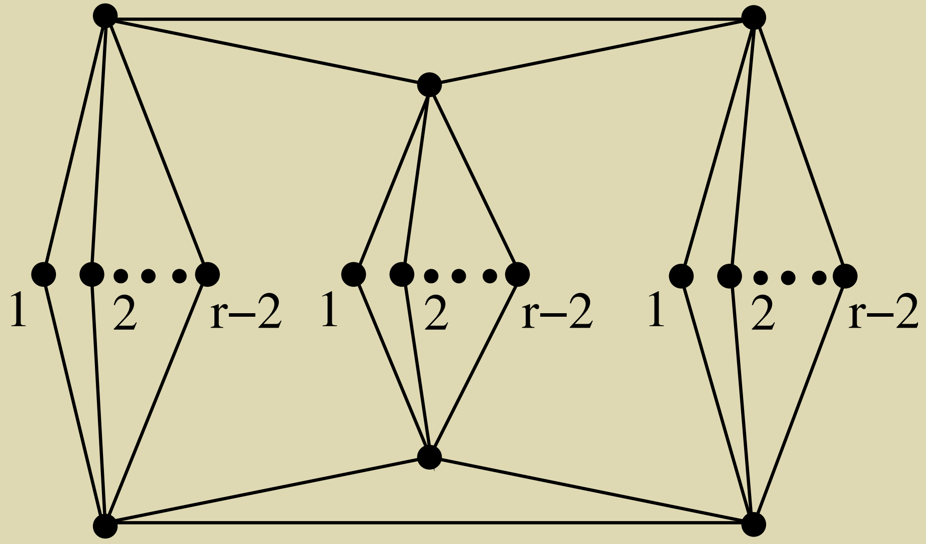
$S_1 \supsetneq S_2 \supsetneq \ldots \supsetneq S_i$.

T' remains 2-connected

$d_{T'}(X) = r - 1$ - contradiction

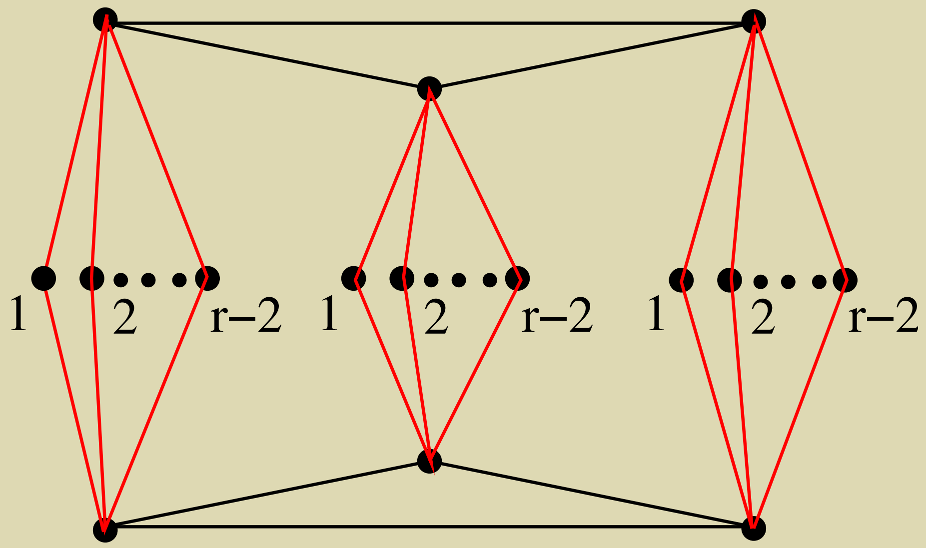


Sharpness of the result



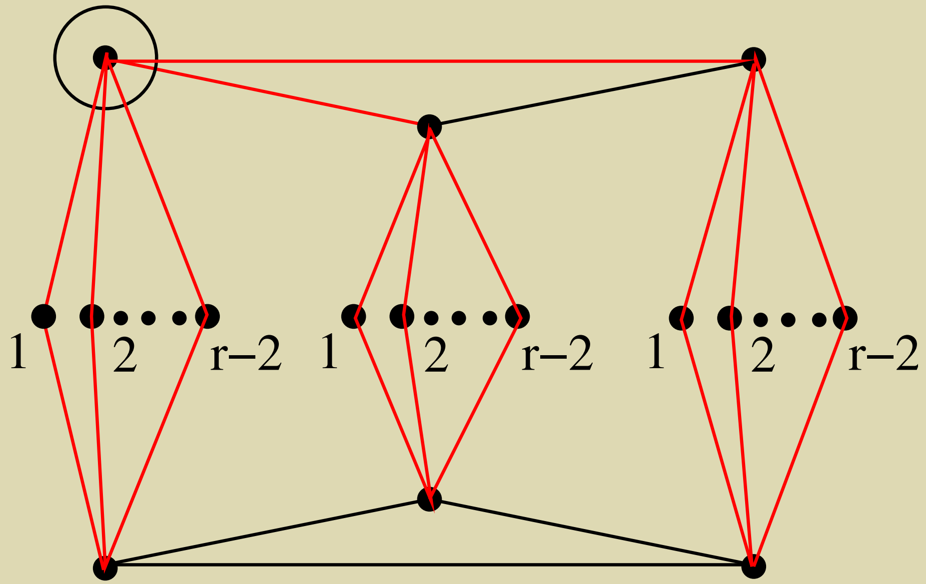
- The example shows a $K_{1,r}$-free graph having an r-trestle but no $(r-1)$-trestle for $r \geq 3$.
- The result of Theorem cannot be improved.

Sharpness of the result



- The example shows a $K_{1,r}$-free graph having an r-trestle but no $(r-1)$-trestle for $r \geq 3$.
- The result of Theorem cannot be improved.

Sharpness of the result



- The example shows a $K_{1,r}$-free graph having an r-trestle but no $(r-1)$-trestle for $r \geq 3$.
- The result of Theorem cannot be improved.

More results on trestles...

- A minimum-degree condition for the existence of an r-trestle was recently proved by Jendrol', Ryjáček and Schiermeyer.
- There is a polynomial algorithm for finding r-trestle in a given $K_{1,r}$-free graph.

More results on trestles...

- A minimum-degree condition for the existence of an r-trestle was recently proved by Jendrol', Ryjáček and Schiermeyer.

- There is a polynomial algorithm for finding r-trestle in a given $K_{1,r}$-free graph.

- Every 2-edge-connected graph with maximum degree Δ has a $\lceil \frac{\Delta+1}{2} \rceil$-walk **(Kaiser, Kužel, Li, Wang; 2006)**

- Every r-trestle has an $\lceil \frac{r+1}{2} \rceil$-walk for any integer $r \geq 2$. **(R. K., J. T. '05)**

The End

Thank You