Compatible Mappings and the Dominating Cycle Conjecture for Snarks

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Joint work with H. Broersma, G. Fijavž, T. Kaiser, R. Kužel, Z. Ryjáček.

Definitions

Graph – loopless (with one exception), multiple edges are allowed, vertex set V, edge set E.

Connected will allways mean edge-connected.

$\boldsymbol{V_k}$**(G)=**$\{x \in V(G) | d_G(x) = k\}$

Cubic graph – every vertex has degree 3, no multiple edges, the graph has at least 5 vertices.

A graph G is **essentially *k*-connected** if G contains no edge cut R such that $|R| < k$ and at least two components of $G - R$ contain at least one edge.

Snark – an essentially 4-connected cubic graph,the length of a shortest cycle is 5.

Dominating cycle (DC) – every edge has at least one vertex on cycle.

Equivalent conjectures

DCC [P.Ash, B. Jackson (1984)] Every essentially 4-connected cubic graph has a DC.

DCC for snarks
[H. Broersma, G. Fijavž, T. Kaiser, R. Kužel, Z. Ryjáček, P.V (2008)]
Every snark has a DC.

Further versions of these equivalent conjectures by **Fleischner (1984), Matthews, Sumner (1984), Thomassen (1985), Kochol** and others.

Contractible techniques

A-contractibility [Ryjáček, Schelp]
no condition on the rest of graph
small class of subgraphs

Weakly A-contractibility
every vertex in subgraph have at most one edge connecting the subgraph with the rest of a graph, one exceptional case
larger class of subgraphs

Compatible mapping
on cubic graphs only, the rest of graph is not edgeless
on cubic graphs most powerful

Cubic fragment – a connected graph without multiple edges with maximum degree 3 and with at least 2 vertices of degree at least 2.

Further extensions need more conditions on the rest of graph or leave connectivity of subgraph.

Compatible mapping

Let F be cubic fragment and let B be a graph with $V(B) \subset V_1(F) \cup V_2(F)$, $E(B) \cap E(F) = \emptyset$ and with components $B_1,...,B_k$.

We say that B is an ***F*-linkage**, if $E(B)$ contains at least one open edge (not loop) and, for any $i = 1,...,k$,

(i) every B_i is a path (of length at least one) or a loop,

(ii) if B_i is a path, then all interior vertices of B_i are in $V_1(F)$,

(iii) if B_i is a loop at a vertex x, then $x \in V_2(F)$.

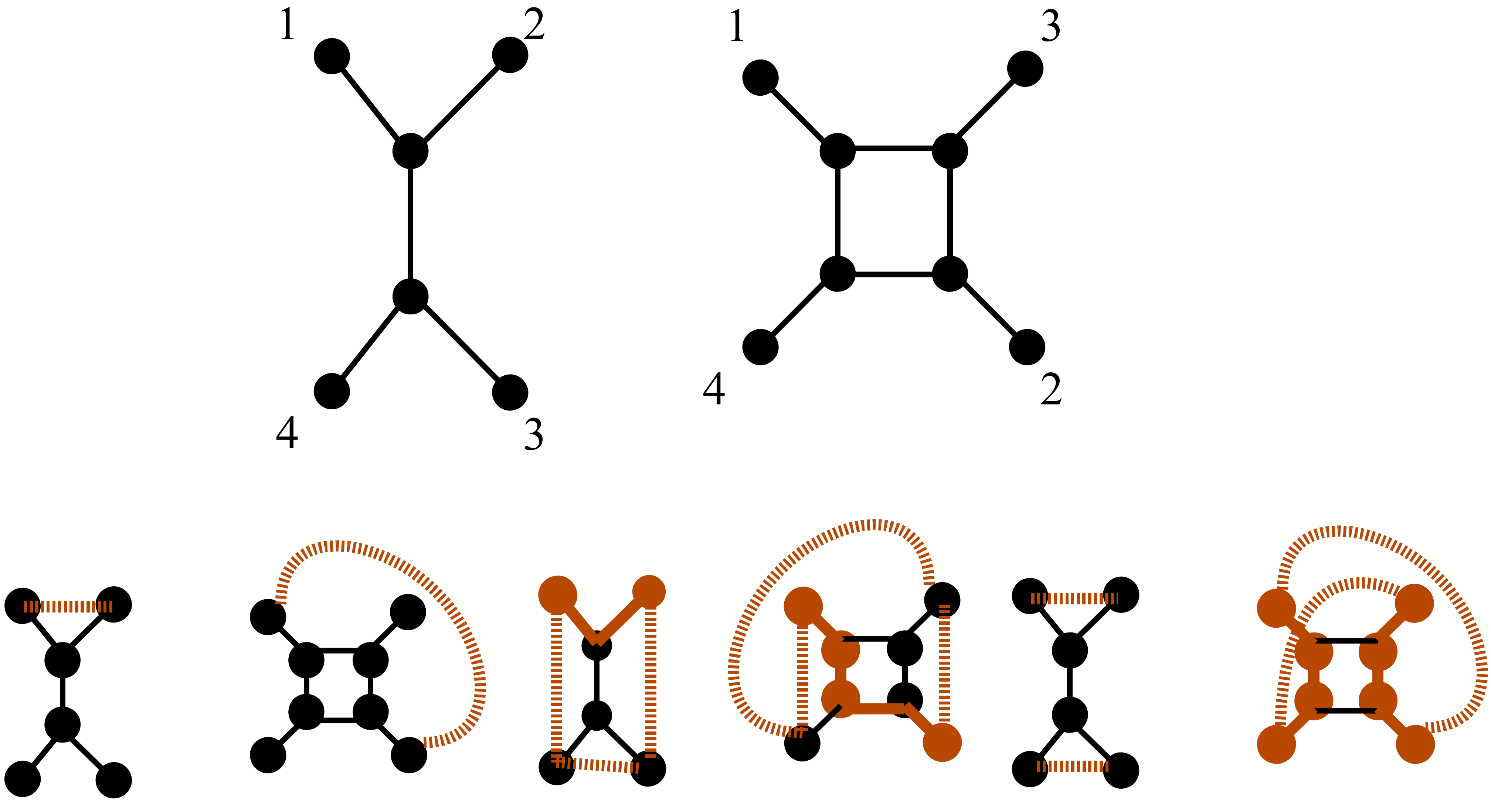
$\boldsymbol{F^B}$ denotes the graph with vertex set $V(F^B) = V(F)$ and edge set $E(F^B) = E(F) \cup E(F^B)$.

Let F_1, F_2 be cubic fragments with $|V_1(F_1) \cup V_2(F_1)| = |V_1(F_2) \cup V_2(F_2)|$ and let $\varphi : V_1(F_1) \cup V_2(F_1) \rightarrow V_1(F_2) \cup V_2(F_2)$ be a bijection. We say that φ is a **compatible mapping** if

(i) $\varphi(V_i(F_1)) = V_i(F_2)$,i = 1,2,

(ii) if B is an F_1-linkage such that F_1^B has a DC containing all open edges of B, then $F_2^{\varphi(B)}$ has a DC containing all open edges of $\varphi(B)$.

Example of a compatible mapping



Theorems for compatible mapping

Theorem Let G be a cubic graph and let C be a DC in G. Let $F \subset G$ be a cubic fragment such that $G - F$ is not edgeless, and let F' be a cubic fragment such that $V(F) \cap V(G) = \varnothing$ and there is a compatible mapping $\varphi : F \to F'$.
Then the graph $G' = G[F \xrightarrow{\varphi} F']$ is a cubic graph having DC C' such that $E(C) \setminus E(F) = E(C') \setminus E(F')$.

Proposition Let X, F be cubic fragments such that there is a compatible mapping $\psi : X \to F$. Let $F_1 \subset F$ be a cubic fragment, and let F_2 be a cubic fragment such that $V(F) \cap V(F_2) = \varnothing$ and there is a compatible mapping $\varphi : F_1 \to F_2$. Let $F' = F[F_1 \xrightarrow{\varphi} F_2]$. Then there is a compatible mapping $\psi' : X \to F'$.

Equivalence of the DCC and DCC for snarks

2-3 graph : every vertex has degree 2 or 3.

Let F be a connected 2-3 graph such that
(i) F contains no cycle of length 4,
(ii) $V_2(F) = 4$,
(iii) F is a subgraph of some essentially 4-connected cubic graph.

Main idea of proof

1. Replacing all C_4 by copies of F, $F \to C_4$ is a compatible mapping.
2. 3-edge-colorable graph $\to$ not 3-edge colorable graph

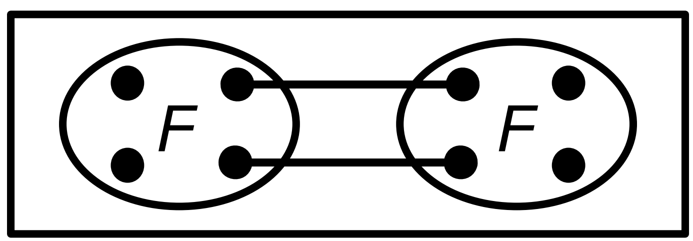
M.Kochol
Equivalence of Fleischner's and Thomassen's conjecture

Does there exist an *F* such that there is a compatible mapping $F \to C_4$?

Reformulating the question

Claim If G is a counterexample of DCC then there is an F which is 2-3 graph, $V_2(F) = 4$ and F is connected subgraph of G such that there is no compatible mapping $C_4 \rightarrow F$.

Claim If there is an F for which vertices of $V_2(F)$ are independent and there is no compatible mapping $C_4 \rightarrow F$ then there is F' for which there is a compatible mapping $F' \rightarrow C_4$.



If DCC doesn't hold,
does there exist an F such that vertices of $V_2(F)$ are independent and there is a compatible mapping $C_4 \rightarrow F$?

Equivalence of the DCC and DCC for snarks

Proposition Let F be a connected 2-3 graph such that

(i) $V_2(F) = 4$,

(ii) F is subgraph of some essentially 4-connected cubic graph,

(iii) there is no compatible mapping $C_4 \rightarrow F$,

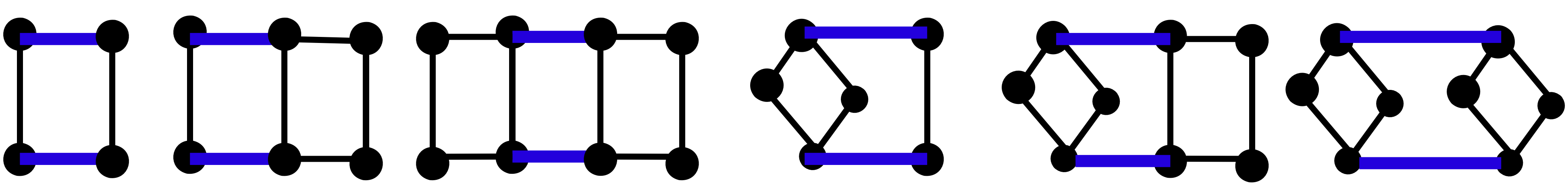
(iv) subject to (i), (ii) and (iii), $|V(F)|$ is minimum.

Then F is essentially 3-connected and contains no cycle of length 4.

Proof.

1. F is essentially 3-connected

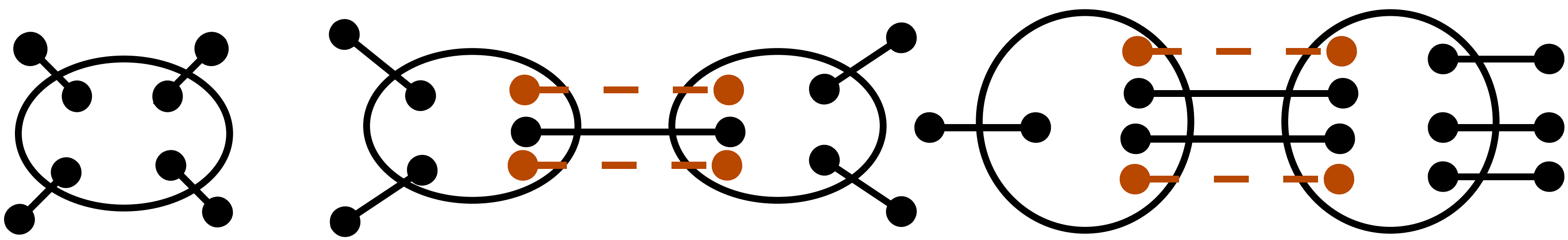
Otherwise, any maximal subgraph H for which there is a compatible mapping $C_4 \rightarrow H$, can be replaced by a C_4.



"Removing" a C_4

2. *F* contains no 2-3 subgraph such that $|V(F)|>4$, $|V_2(F)|=4$.
3. *F* contains no cycle of length 4.

we replace C_4 -> edge (the replacement is possible since *F* contains no triangle) and we still have a graph which fulfils the conditions.



Counterexample to the conjecture 3 -> there is an *F* (by the claim) -> counterexample to the DCC for snarks (by the Kochol's construction (edge coloring))

Equivalent conjectures

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Thank you for your attention

Questions ?