Some Results on Paths and Cycles in Claw-Free Graphs

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1. Basic Concepts

A graph G is called claw-free if it has no induced subgraph isomorphic to $K_{1,3}$.

A line graph $L(G)$ of a graph G is a graph in which $V(L(G)) = E(G)$ and where two vertices are adjacent if and only if they are adjacent as edges of G.

A graph is hamiltonian if there exists a cycle containing every vertex of G.

A graph is Hamilton-connected if any pair of vertices is joined by a hamiltonian path.

For convenience, let $H_u = G[N(u)]$. A vertex u of G is said to be locally connected if H_u is connected.

2. Hamiltonicity

Conjecture 2.1(Matthews & Sumner, JGT, 1983). Every 4-connected claw-free graph is hamiltonian.

Since every line graph is claw-free, the following conjecture proposed in 1986 by Thomassen is a special case of Conjecture 2.1.

Conjecture 2.2 (Thomassen,). Every 4-connected line graph is hamiltonian.

An important progress on Conjecture 2.2 is due to Zhan(Dis. Math., 91) and independently to Jackson.

Theorem 2.3 (Zhan; Jackson). Every 7-connected line graph is hamiltonian.

Some important progresses

The Ryjáček's closure of a claw-free graph G, denoted by $cl_R(G)$, is obtained from G by successively adding all missing edges to the neighborhood of a locally connected vertex.

Theorem 2.4 (Ryjáček). Let G be a claw-free graph. Then there is a triangle-free graph H such that $cl_R(G) = L(H)$ and $c(cl_R(G)) = c(G)$.

It follows from Theorem 2.3 and Theorem 2.4 that every 7-connected claw-free graph is hamiltonian.

Theorem 2.5 (Li). Every 6-connected claw-free graph with at most 33 vertices of degree 6 is hamiltonian.

Theorem 2.6 (Fan). Every 6-connected claw-free graph with all vertices of degree 6 independent is hamiltonian.

Hu, Tian and Wei got the following two theorems.

Theorem 2.7. Let G be a 6-connected claw-free graph and let $V_0 = \{ v \in V(G) : d_G(v) = 6 \}$. If $|V_0| \leq 44$ or $G[V_0]$ contains at most 8 vertex disjoint K_4's, then G is hamiltonian.

Clearly, Theorem 2.7 is a generalization of Theorems 2.3, 2.5 and 2.6.

3. Hamiltonian Connectivity

For Hamilton-connectedness of claw-free graphs, no constant connectivity bound for it was known, until Brandt got the following striking result:

Theorem 3.1 (Brandt, JCTB, 99). Every 9-connected claw-free graph is Hamilton-connected.

By considering the line graph, Hu, Tian and Wei obtained:

Theorem 3.2. Let G be a 6-connected line graph and let $V_0 = \{\, v \in V(G) : d_G(v) = 6 \,\}$. If $|V_0| \leq 29$ or $G[V_0]$ contains at most 5 vertex disjoint K_4's, then G is Hamilton-connected.

By using the closure idea of Brandt and Theorem 3.2, Hu, Tian and Wei got following result.

Theorem 3.3. Let G be a 7-connected claw-free graph and let a and b be any two distinct vertices of G. If $\{a, b\}$ is not contained in any vertex cut of order 7 of G, then G has a hamiltonian (a, b)-path.

Theorem 3.3 has the following corollary.

Corollary 3.4. Every 8-connected claw-free graph is Hamilton-connected.

4. Ideas of the proof of Th. 3.2

More notations and definitions

If A and B are subgraphs of G or subsets of $V(G)$, we define $M_G(A) = \{ e \in E(G) : \ e$ has only one end vertex in $A\}$ and $M_G(A, B) = M_G(A) \cap M_G(B)$. The cardinalities of $M_G(A)$ and $M_G(A, B)$ are denoted by $m_G(A)$ and $m_G(A, B)$, respectively. In particular, when $A = \{x\}$ and $B = \{y\}$, we set $M_G(x) = M_G(\{x\})$ and define $m_G(x)$, $M_G(x, y)$ and $m_G(x, y)$ similarly.

Notice that $m_G(x, y)$ is the number of multiple edges between x and y.

For $\emptyset \neq S \subset V(G)$, let $G[S]$ denote the subgraph of G induced by S and define $G - S = G[V(G) - S]$.

If E_0 is a subset of the edge set $E(G)$, then we use $G - E_0$ to denote the spanning subgraph with $V(G - E_0) = V(G)$, $E(G - E_0) = E(G) - E_0$. For a vertex $x \in V(G)$, define $N_G(x) = \{ y \in V(G) : xy \in E(G) \}$.

If X is a subset of V_0 such that $G[X] \cong K_4$, then we call $G[X]$ a bad K_4 of G. A vertex v is called a bad vertex if it is a vertex of a bad K_4 of G. Let $b(G)$ be the number of bad vertices of G. Define

$$\mu(G) = \max\{h : G[V_0] \text{ has } h \text{ vertex disjoint } K_4\text{'s} \}.$$

A set $D \subseteq V(G)$ is called a dominating set of G if every edge of G has at least one end vertex in D (i.e., $E(G - D) = \emptyset$).

A graph G is essentially k-edge-connected if $|E(G)| \geq k+1$ and $G - E_0$ has exactly one component H with $E(H) \neq \emptyset$ for all $E_0 \subseteq E(G)$, $|E_0| < k$.

A trail in G is a finite sequence of vertices and distinct edges

$$T = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$$

such that e_i, $1 \leq i \leq k$, is an edge in G with end vertices v_i and v_{i+1}. If, in addition, $v_1 = v_{k+1}$, T is called a closed trail.

The internal vertices of T are v_i for $2 \leq i \leq k$ when $v_1 \neq v_{k+1}$ and every vertex of T is an internal vertex for a closed trail T.

A trail in G is a dominating trail if each edge of G is incident with at least one internal vertex of the trail.

A trail in G is a spanning trail if for each vertex v of G there exists an internal vertex v_i of the trail such that $v_i = v$.

A graph G is dominating trailable if for each pair e_1 and e_2 of edges of G there is a dominating trail $e_1 T_d e_2$ with end edges e_1 and e_2.

A graph G is spanning trailable if for each pair e_1 and e_2 of edges of G there is a spanning trail $e_1 T_s e_2$ with end edges e_1 and e_2.

A graph H is called a multi-star if it is obtained from some star $K_{1,s}$ by adding some multiple edges incident with the center.

Assume that H is not a multi-star.

Lemma 4.1. Let H be a graph such that $L(H)$ is k-connected. Then

(i) $m_H(x,y) \leq max\{\frac{1}{2}(d_H(x)+d_H(y)-k),\ 0\}$, for all $x, y \in V(H)$ with $x \neq y$ and $d_H(x)+d_H(y) \leq k+2$.

(ii) $D' = \{v \in V(H) :\ d_H(v) \geq \frac{k+2}{2}\}$ is a dominating set of H.

(iii) If there exists a pair (A, B) with $A \subseteq \{v \in V(H) :\ d_H(v) = 5\}$ and $B \subseteq \{v \in V(H) :\ d_H(v) = 3\}$ such that $m_H(A,B) > 3|A| + \min\{\frac{2}{5}b(L(H)),\ 2\mu(L(H))\}$, then $\kappa(L(H)) < 6$.

Two operations

R_1: delete a vertex u, which has degree at most 2 but is adjacent to at most one vertex, and delete its incident edges;

R_2: delete a vertex u with degree 2 and its incident edges uv and uw, where $v \neq w$, and add a new edge vw.

Use the above two operations we can simplify the graph.

Lemma 4.2. Let H be a graph, which is not a multi-star of edge multiplicity at most 5 and, let its line graph $L(H)$ be 6-connected. Then, there is a unique graph (up to isomorphism) H^*, called the reduced graph of H, obtained from H by applying a sequence of operations R_1 and R_2 such that:

(i) $\delta(H^*) \geq 3$;

(ii) $\kappa(L(H^*)) = \kappa(L(H)) \geq 6$;

(iii) $M_{H^*}(x) = M_H(x)$, for any $x \in V(H^*)$ with $d_{H^*}(x) < 6$;

(iv) $V(H^*) = D(H^*) = D(H)$ is a dominating set of H;

(v) $m_{H^*}(A, B) = m_H(A, B) \leq 3|A| + \min\{\frac{2}{5}b(L(H)), 2\mu(L(H))\}$ for each pair (A, B) with $A \subseteq \{\, v \in V(H^*) : \ d_{H^*}(v) = 5 \,\}$ and $B \subseteq \{\, v \in V(H^*) : \ d_{H^*}(v) = 3 \,\}$.

Lemma 4.3. Let H be a graph, which is not a multi-star of edge multiplicity at most 5, and let its line graph $L(H)$ be 6-connected. Then, H is dominating trailable if its reduced graph H^* is spanning trailable.

Lemma 4.4. If H is a graph with at least 4 vertices, then its line graph $L(H)$ is Hamilton-connected if and only if H is dominating trailable or H is isomorphic to a multi-star.

Lemma 4.4 can be proved by a slight modification of the proof of the following theorem.

Theorem 4.5(Harary & Nash-Williams). If H is a graph with at least 4 vertices, then its line graph $L(H)$ is hamiltonian if and only if H has a dominating closed trail or H is isomorphic to $K_{1,s}$, for some integer $s \geq 3$.

Theorem 4.6(Nash-Williams; Tutte). A graph G has k edge-disjoint spanning trees if and only if

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

Theorem 4.7. Let G be a graph such that

$$|E_0| \geq 2\omega(G - E_0) - 1$$

for all $E_0 \subseteq E(G)$, $E_0 \neq \emptyset$. Then G is spanning trailable.

Ideas of the proof of Th. 4.7

Let e_1 and e_2 be any two edges of G, We show

1. $G - \{e_1\}$ two edge-disjoint spanning trees, say T_1 and T_2.(By Theorem 4.?).

2. Assume that $e_2 \in E(T_1) \cup E(T_2)$, say $e_2 \in E(T_1)$ and let X be the set of vertices with odd degrees in T_1. Then $|X|$ must be even. Set $X = \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$. For $1 \le i \le k$, let P_i be the path joining x_i and y_i in T_2. Define $E_1 = E(P_1)$ and for $2 \le j \le k$ let $E_j = (E(P_j) - E_{j-1}) \cup (E_{j-1} - E(P_j))$. Then $E_j \subseteq E(T_2)$ for any $1 \le j \le k$. It is easily seen that the set of vertices with odd degrees in the graph $G[E_j]$ is $\{ x_1, y_1, \cdots, x_j, y_j \}$. Since $E(T_1) \cap E_k = \emptyset$, $G[E(T_1) \cup E_k]$ must be

eulerian. Regarded as a trail, $E(T_1) \cup E_k$ is a spanning closed trail of G, containing e_2 but not containing e_1. Denote this trail by $T_s = v_2, e_2, v_3, e_3, \ldots, v_t, e_t, v_2$.

3. Add e_1 to T_s to find a spanning trail of G with end edges e_2 and e_1.

Lemma 4.8. Let H be a graph which is not a multi-star of edge multiplicity at most 5. If E_0 is a subset of $E(H^*)$ with $E_0 \neq \emptyset$, then

$|E_0| \geq 2\omega(H^* - E_0) - 1$, if $\omega(H^* - E_0) \leq 2$;

$|E_0| \geq 2\omega(H^* - E_0)$
$- \min\{\frac{1}{15}b(L(H)), \frac{1}{3}\mu(L(H))\}$, otherwise.

Ideas of the proof

Let $\omega = \omega(H^* - E_0)$. It is easy to verify that Lemma 4.8 is true if $\omega \leq 2$ by Lemma 4.2. So we assume that $\omega \geq 3$. Let $H_1, H_2, \ldots, H_\omega$ be all the components of $H^* - E_0$. It follows from Lemma 4.2 that H^* is essentially 6-edge-connected and $\delta(H^*) \geq 3$. Define

$S_j = \{ i : 1 \leq i \leq \omega, m_{H^*}(H_i) = j \}$, $3 \leq j \leq 5$

and

$S_6 = \{ i : 1 \leq i \leq \omega, m_{H^*}(H_i) \geq 6 \}$.

Let $s_j = |S_j|$, $3 \leq j \leq 6$. Then we have $\omega(H^* - E_0) = s_3 + s_4 + s_5 + s_6$ and

$$|E_0| \geq \frac{1}{2}(3s_3 + 4s_4 + 5s_5 + 6s_6). \qquad (1)$$

Furthermore, we have $|V(H_i)| = 1$ for all $i \in S_3 \cup S_4 \cup S_5$. For $3 \leq j \leq 6$, let $X_j = \cup_{i \in S_j} V(H_i)$. Then, $|X_j| = s_j$, $3 \leq j \leq 5$. By Lemma 1(i), we have $E(H^*[X_3]) = \emptyset$ and $M_{H^*}(X_3, X_4) = \emptyset$. Thus, $2|E(H^*[X_4 \cup X_5])| \leq 4|X_4| + 5|X_5| - m_{H^*}(X_5, X_3)$. Counting the edges of E_0 that have at least one end in $X_3 \cup X_4 \cup X_5$, we have

$|E_0| \geq$
$3|X_3| + 4|X_4| + 5|X_5| - m_{H^*}(X_3, X_4) - |E(H^*[X_4 \cup X_5])| - m_{H^*}(X_5, X_3)$
$\geq 3|X_3|+4|X_4|+5|X_5|-\frac{1}{2}(4|X_4|+5|X_5|- m_{H^*}(X_5, X_3)) - m_{H^*}(X_5, X_3)$
$= 3s_3 + \frac{1}{2}(4s_4 + 5s_5) - \frac{1}{2}m_{H^*}(X_5, X_3).(3)$

By (2) and (3), we get

$3|E_0| \geq 6(s_3 + s_4 + s_5 + s_6) + \frac{1}{2}(3s_5 - m_{H^*}(X_5, X_3)).$

This together with $|X_5| = s_5$ and Lemma 4.2(v) implies

$3|E_0| \geq$
$6\omega(H^*-E_0)-\frac{1}{2}\min\{\frac{2}{5}b(L(H))),\ 2\mu(L(H)\}.$

Therefore,

$|E_0| \geq$
$2\omega(H^*-E_0)-\min\{\frac{1}{15}b(L(H))),\ \frac{1}{3}\mu(L(H))\}.$

5. One possible approach

To attack the conjecture 1.2 for a 6-connected line graph G, we first investigate the following:

Problem: If H is essentially 6-edge-connected with $\Delta(H) \leq 5$, does G have a dominating closed trail?

If the answer for the above problem is posithenive, the we can do some operstions for graphs with maximum degree at least 6 to keep the new graph being still essentially 6-edge-connected and study the relationship between these two graph.