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On 2-Factors of Claw-free Graphs

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$n := |V(G)|$

δ : **the minimum degree of** G

Conjecture 1 (Fujisawa, Xiong, Y, Zhang 2007)**.** If G is a graph with $\delta \geq 3$,
$\Longrightarrow$ its line graph has a 2-factor with at most

$$\frac{(2\delta - 3)n}{2(\delta^2 - \delta - 1)} \quad (< \frac{n}{\delta}) \text{ cycles.}$$

Def 1.

1. **Line graph** $L(G)$: the vertex set is $E(G)$ and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.
2. F is a **2-factor** of $G \Longleftrightarrow F$ is a spanning 2-regular graph of G.

A hamilton cycle is a **connected** 2-factor.

Conjecture 2 (Thomassen 1984)**.**

A 4-connected line graph is hamiltonian.

Conjecture 3 (Matthews and Sumner 1986)**.**

A 4-connected claw-free graph is hamiltonian.

- A graph is called **claw-free**

  if G has no induced $K_{1,3}$.

- A line graph is a claw-free graph.

Thm 1 (Ryjecek 1997)**.**

Conjecture 2 and Conjecture 3 are equivalent.

Thm 2 (Zhan + Thm 1)**.**

A 7-connected claw-free graph is hamiltonian.

Our question is that

> What will happen if the connectivity of a claw-free graph is smaller than 4?

Of course, it is false that

> Every 3-connected claw-free graph is hamiltonian.

However, the following statement holds.

Thm 3 (Egawa and Ota 1991, Choudum and Paulraj 1991)**.**
Let G be a claw-free graph and l be any positive integer at most $\delta/2$.

$$\text{If } ln \text{ is even } \implies G \supset l\text{-factor.}$$

Cor 4. If G is a claw-free graph with $\delta \geq 4$, then G has a 2-factor.

How many cycles does a 2-factor of a claw-free graph have?

Thm 5 (Faudree et al. 1999)**.**
Every claw-free graph G with $\delta \geq 4$ has a 2-factor with at most

$$\frac{6n}{(\delta+2)} - 1 \text{ cycles.}$$

Thm 6 (Gould and Jacobson 2001)**.**
Every claw-free graph G with $\delta \geq (4n)^{\frac{2}{3}}$ has a 2-factor with at most

$$\frac{n}{\delta} \text{ cycles.}$$

Fact 7 (Y 2007).

There exists an infinite family of line graphs G with $\delta \geq 4$ in which every 2-factor contains more than

$$\frac{n}{\delta} \text{ cycles.}$$

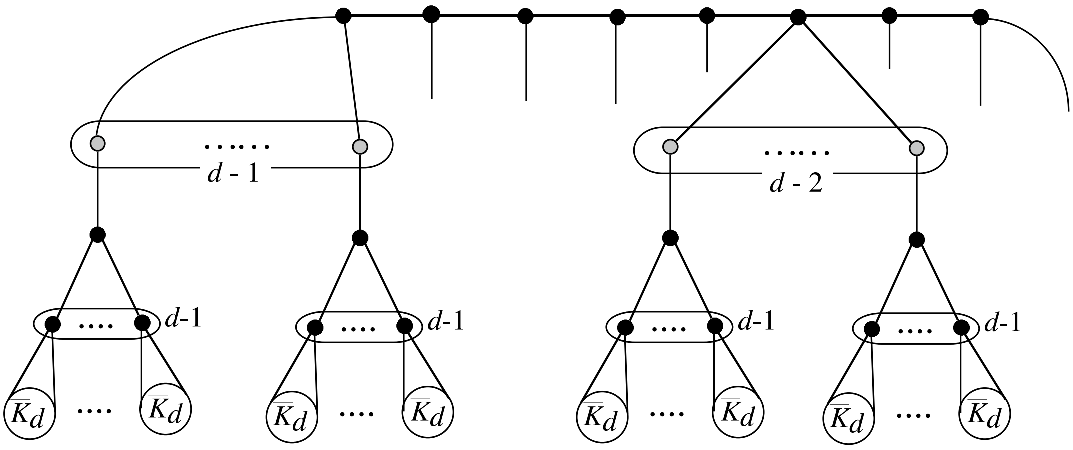

Figure 1: $H^*_{m,d}$

Any 2-factor of the line graph $L(H^*_{m,\delta})$ contains

$$\frac{(\delta^2 - 2\delta + 1)n - (\delta^2 + 1)}{\delta^3 - 2\delta^2 + \delta - 1} \; (> \frac{n}{\delta}) \text{ cycles.}$$

Fact 8 (Y 2007)**.**

There exists a family $\{G_i\}$ of line graphs with $\delta \geq 3$ such that

$$\frac{f_2(G_i)}{|G_i|} \to \frac{5}{18} \quad (|G_i| \to \infty),$$

where $f_2(G)$ is the minimum number of components in a 2-factor of G.

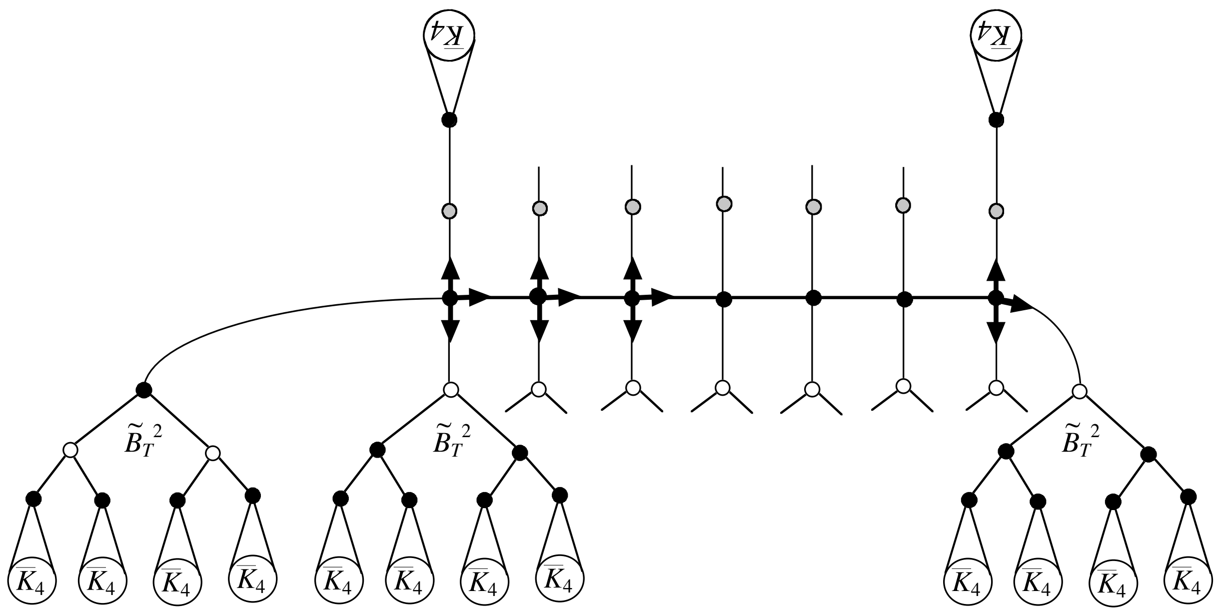

Figure 2: $B_{m,2k}$

Problem 4 (Y 2007).

1. Does every claw-free graph with $\delta = 4$ have a 2-factor with at most
$$\frac{5n}{18} \text{ cycles?}$$

2. Does every claw-free graph with $\delta \geq 5$ have a 2-factor with less than
$$\frac{n}{\delta - 1} \text{ cycles?}$$

Thm 9 (Broersma, Paulusma and Y)**.**

1. A claw-free graph with $\delta = 4$ has a 2-factor with at most

$$\frac{5n-14}{18} \text{ cycles.}$$

2. A claw-free graph with $\delta \geq 5$ has a 2-factor with at most

$$\frac{n-3}{\delta-1} \text{ cycles.}$$

Def 2. Let G be a claw-free graph.

If, for a vertex $x \in G$

- $G[N(x)]$ is connected and
- there are non-adjacent vertices in $G[N(x)]$,

then we add edges joining all pairs of non-adjacent vertices in $N(x)$.

The **Ryjacek closure** $cl(G)$ of G is a graph obtained by recursively repeating this operation, as long as this is possible.

Thm 10 (Ryjáček 1997).

- The closure of a claw-free graph G is uniquely determined.
- There exists a triangle-free graph H such that $L(H) = cl(G)$.

Thm 11 (Ryjáček 1997).

A claw-free graph G is hamiltonian

$\iff cl(G)$ is hamiltonian.

Thm 12 (Ryjáček, Saito and Shelp 1999).

If G is a claw-free graph, then

$$f_2(G) = f_2(cl(G)),$$

where $f_2(G)$ is the minimum number of components in a 2-factor of G.

Theorem A (Harary and Nash-Williams 1965)**.**

The line graph $L(G)$ has a Hamilton cycle

$\iff G$ has a dominating circuit.

$H \subset G$ is **dominating** if $G - H$ *is* edgeless.

Def 3. A set $\mathcal{S}$ of

circuits and stars with at least 3 edges

is called k-**system** if

every edge $e \in E(G) - \bigcup_{C \in \mathcal{S}} E(C)$ is incident to a circuit in $\mathcal{S}$.

Prop 13 (Gould and Hynds 1999)**.**
A graph H has a k-system
$\Longleftrightarrow$ $L(H)$ has a 2-factor with k components.

Cor 14.
A claw-free graph G has a 2-factor with k cycles
$\Longleftrightarrow$ the preimage graph H of G has a k-system.

Thm 15 (Y 2007)**.**

A 2-connected claw-free graph with $\delta \geq 3$ has a 2-factor.

Lem 16.

An essentially 2-edge-connected graph with minimum edge-degree $\delta_E \geq 3$ has a k-system.

Thm 17 (Petersen 1891)**.**

A bridge-less cubic multigraph has a 1-factor.

Cor 18.

A bridge-less cubic multigraph has a 2-factor.

Thm 19 (Fleischner 1992)**.**

A bridge-less multigraph with $\delta \geq 3$ has an even-factor.

An **even-factor** F of G is a spanning subgraph of G such that $d_F(v) \equiv 0 \pmod 2$.

Let H be an essentially 2-edge-connected graph with $\delta_E \geq 3$.

$\Longrightarrow$ $V_{\leq 2}(H)$ is an independent set.

— Suppose $V_1 = \emptyset$.

We suppress all vertices of degree 2.

$\Longrightarrow$ The minimum degree of the remaining graph R is at least 3.

$\Longrightarrow$ By Fleischner's theorem,

R has an even-factor F.

The subgraph in G corresponding to F is k-system of G, which is constructed by circuits.

Conjecture 5 (Y 2007)**.**

If G is a bridge-less claw-free graph,

$\Longrightarrow G$ has a 2-factor with at most

$$\frac{n-1}{\delta} \text{ cycles.}$$

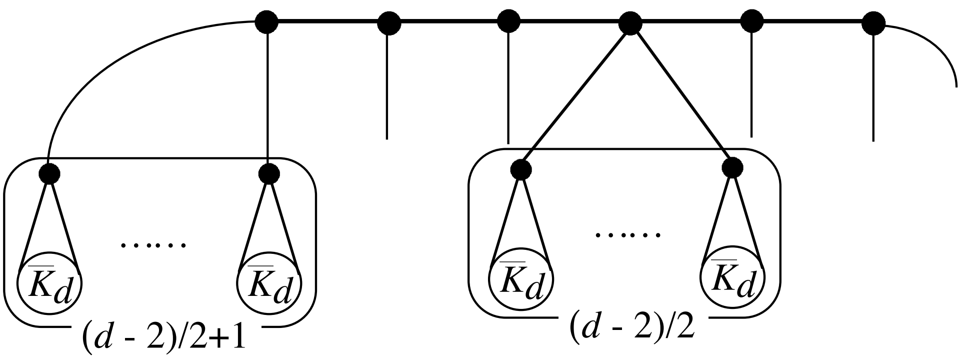

Figure 3: $H_{2m,d}$

Thm 20 (Jackson and Y 2007)**.**

1. A 2-connected claw-free graph has a 2-factor with at most

$$\frac{n+1}{4} \text{ cycles.}$$

2. A 3-connected claw-free graph has a 2-factor with at most

$$\frac{2n}{15} \text{ cycles.}$$

Lem 21.

1. An essentially 2-edge-connected graph has a k-system such that
$$k \leq \frac{n+1}{4}$$
2. An essentially 3-edge-connected graph has a k-system such that
$$k \leq \frac{2n}{15}$$

For proving these theorems,

we modify Fleischner's theorem.

Thm 22 (Jackson and Y 2007)**.**

1. A bridge-less graph with $\delta \geq 3$ has an even-factor in which each component contains at least 4 vertices.

2. A 3-edge-connected multigraph with $\delta \geq 3$ has an even-factor in which each component contains at least 5 vertices.

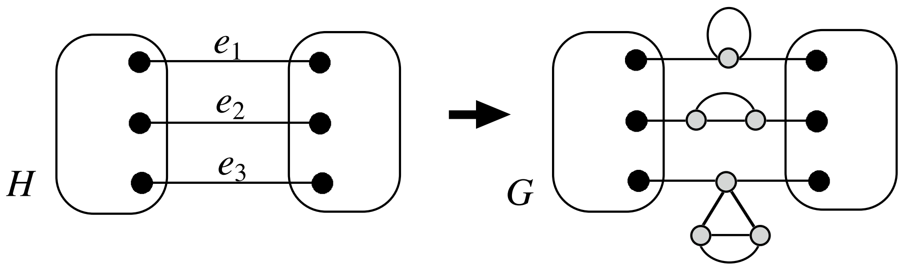

Figure 4:

We prove at first

Cubic graphs has the required property.

And next,

We use the vertex splitting operation to reduce to the cubic case.

Thm 23 (Jackson and Y 2007)**.**
If G is a graph with $\delta \geq 3$,
$\Longrightarrow$ its line graph has a 2-factor with at most

$$\frac{3n}{10} \text{ cycles.}$$

Thm 24.

There exists an infinite family of essentially 4-edge-connected cubic graphs in which every 2-factor has a 5-cycle.

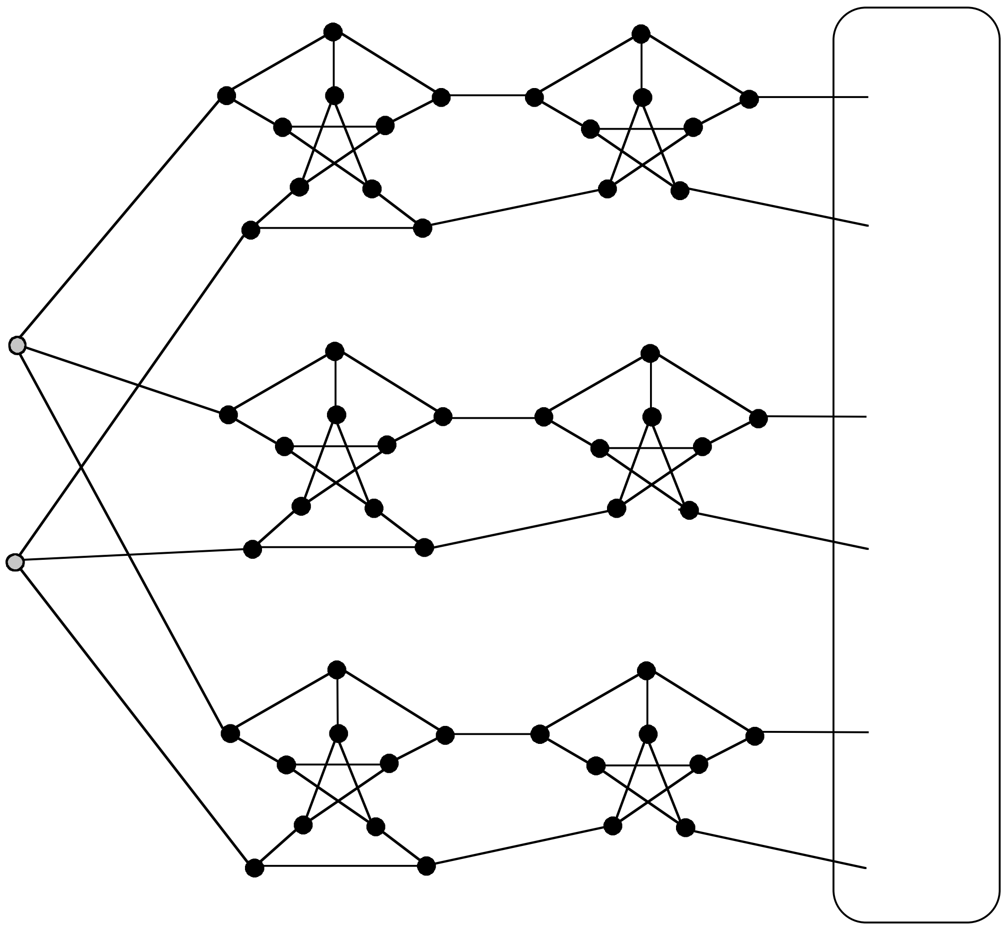

Figure 5:

Thm 25 (Kochol 2002)**.**
There exists an infinite family of cyclically 6-edge-connected cubic graphs in which, for any 2-factor X has at least

$$\frac{n}{118} \text{ components.}$$

Problem 6 (Jackson and Y)**.**
Is there a value of k for which there exist an unbounded function $g : \mathbb{N} \to \mathbb{N}$ such that every cyclically k-edge-connected cubic graph G has a 2-factor X in which every cycle has at least $g(n)$ vertices?

By the way, can we find a property of a graph with $\delta \geq 3$ such that

> for cubic graphs, the property hold trivially?

For instance,

Conjecture 7.

Every essentially 4-edge-connected graph has circuit passing through all vertices of degree at least 4.