Hourglasses and Hamilton cycles in 4-connected claw-free graphs

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Abstract

We show that if G is a 4-connected claw-free graph in which every induced hourglass subgraph S contains two non-adjacent vertices with a common neighbor outside S, then G is hamiltonian. This extends the fact that 4-connected claw-free, hourglass-free graphs are hamiltonian, thus proving a broader special case of a conjecture by Matthews and Sumner.

1 Introduction

A well-known conjecture of Matthews and Sumner (see [7]) states that all 4-connected claw-free graphs are hamiltonian. (A graph is *claw-free* if it contains no *claw*, that is, no induced $K_{1,3}$.) While the conjecture is still wide open, it has been proved in various special cases. One such result concerns *induced hourglasses*, i.e.,

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induced subgraphs isomorphic to the graph in Fig. 1a: if a 4-connected claw-free graph is *hourglass-free* (contains no induced hourglass), then it is hamiltonian. This was observed independently by several authors; see, e.g., [1].

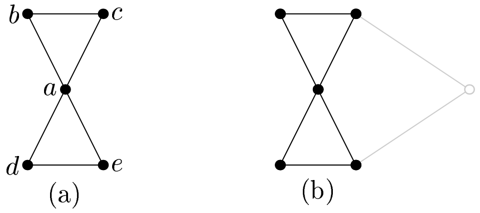

Figure 1: (a) An hourglass. (b) A common neighbor as in the hourglass property.

In the present paper, we generalize this result to graphs which may contain some induced hourglasses. A graph G has the *hourglass property* if in every induced hourglass S, there are two non-adjacent vertices which have a common neighbor in $G - V(S)$ (see Fig. 1b). We prove:

Theorem 1 *Every 4-connected claw-free graph with the hourglass property is hamiltonian.*

2 Notation

Let us fix some notation. The induced subgraph G' of a graph G on vertices $V(G') = \{ a_1, a_2, \ldots, a_k \}$ is denoted by $\langle a_1 a_2 \ldots a_k \rangle_G$. The order of the vertices has a special meaning whenever we speak of an induced hourglass: $\langle abcde \rangle_G$ is an hourglass centered at a and containing edges bc and de (such as the one in Fig. 1). We use a similar convention for claws: by saying that $\langle abcd \rangle_G$ is a claw, we always mean that a is the center.

The *neighborhood* of a vertex v in G is denoted by $N_G(v)$. (Note that $v \notin N_G(v)$.) With a slight abuse of notation, we also write $N_G(v)$ for the induced subgraph $\langle N_G(v) \rangle_G$. The vertex v is *locally connected* if $N_G(v)$ is connected. Otherwise, v is *locally disconnected.*

3 Stability

We need the concept of the *claw-free closure* as defined in [9]. Let G be a claw-free graph. If x is a locally connected vertex of G, then the *local completion at* x is the operation of adding all possible edges between vertices in $N_G(x)$. The resulting graph, denoted by G'_x, is easily shown to be claw-free again. Iterating local completions, we finally arrive at a graph in which all locally connected vertices have complete neighborhoods. This graph does not depend on the order of local completions (which is not quite obvious); it is called the *closure* of G and denoted by $cl(G)$.

Theorem 2 (Ryjáček [9]) *For a claw-free graph G,*

(i) the (well-defined) closure $cl(G)$ of G is the line graph of some triangle-free graph,

(ii) the closure is idempotent, i.e. $cl(cl(G)) = cl(G)$,

(iii) G is hamiltonian if and only if $cl(G)$ is hamiltonian.

The following proposition will enable us to restrict our attention to line graphs.

Proposition 3 *Let G be a claw-free graph having the hourglass property. Then its closure $cl(G)$ has the hourglass property, too.*

The proof of Proposition 3 is given below. First note that a local completion of an hourglass-free graph G may contain induced hourglasses. However, as shown in [2], all such hourglasses are destroyed by subsequent local completions, so the closure $cl(G)$ is hourglass-free. This motivates the following definition.

An induced hourglass in G is *permanent* if its vertex set induces an hourglass in $cl(G)$. The graph G has the *permanent hourglass property* if for every permanent induced hourglass $S \subset G$, some two non-adjacent vertices of S have a common neighbor in $G - V(S)$. Thus the permanent hourglass property is just the hourglass property restricted to permanent hourglasses. We shall need two lemmas about the closure.

Lemma 4 *Let v be a vertex of a claw-free graph G. If u is any internal vertex of an induced path in $N_G(v)$, then $N_{cl(G)}(u)$ is complete.*

Proof. Follows easily from the observations that u must be a locally connected vertex in G, and that local completions cannot make any locally connected vertex locally disconnected. □

Lemma 5 *An induced hourglass $H = \langle ab_1b_2c_1c_2\rangle_G$ in a claw-free graph G is permanent if and only if $N_{cl(G)}(a)$ contains no b_ic_j-path.*

Proof. Assume that H is permanent. If $N_{cl(G)}(a)$ contains a b_ic_j-path, then the edges b_1b_2 and c_1c_2 are in the same component of $N_{cl(G)}(a)$. Since each component of $N_{cl(G)}(a)$ is clearly a complete subgraph, H cannot be an induced hourglass.

For the converse, note that an edge joining b_i and c_j is a particular example of a b_ic_j-path. If there are no such edges, then H is an induced subgraph of $cl(G)$ and is therefore permanent. □

Proposition 6 *If a claw-free graph G has the permanent hourglass property, then so does its local completion G'_x at any locally connected vertex x.*

Proof. Suppose that $H = \langle ab_1b_2c_1c_2\rangle_{G'_x}$ is a permanent hourglass in G'_x such that no b_i has a common neighbor with any c_j except a. Trivially, $H^- = \langle ab_1b_2c_1c_2\rangle_G$ cannot be an induced hourglass, for it would be permanent and the above common neighbor would have to exist.

Thus, some edges of H are missing in H^-. At least one edge adjacent to a must be missing. To see this, note that if $b_1b_2 \notin E(H^-)$ and H^- contains all the edges adjacent to a, then $\langle ab_1b_2c_1\rangle_G$ is a claw. By symmetry, we may henceforth assume that $ab_1 \notin E(H^-)$. Since the local completion at x in G adds the edge ab_1, we have $xa, xb_1 \in E(G)$. Since b_1 is non-adjacent in G'_x to c_i (for $i = 1, 2$), and $ac_i \in E(G'_x)$, we can conclude that $ac_i \in E(G)$. The same argument proves that $c_1c_2 \in E(G)$.

Choose a shortest ab_1-path P in $N_G(x)$ and let p be the neighbor of a on P. (Refer to Fig. 2 for an illustration.) We claim that $J = \langle axpc_1c_2\rangle_G$ is an hourglass. If not, then either xc_i or pc_i is an edge for some i. In either case, we get a b_1c_i-path in $N_{G'_x}(a)$ and hence in $N_{cl(G)}(a)$; these paths are b_1xc_i and b_1xpc_i, respectively. By Lemma 5, H is not permanent. This contradiction implies that J is an hourglass. In fact, J is a permanent hourglass: if a subsequent local completion destroys J, the added edge (xc_i or pc_i) creates a b_1c_i-path in $N_{cl(G)}(a)$.

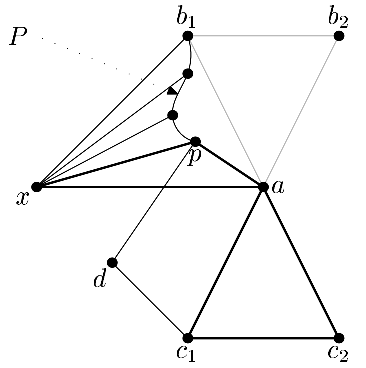

Figure 2: An illustration to the proof of Proposition 6. 'Guaranteed' edges of G are shown black, those of G'_x grey, the hourglass J bold.

By the permanent hourglass property of G, either x or v has a common neighbor d with some c_i. If d is a common neighbor of x and c_i, then $b_1d \in E(G'_x)$ and d is a common neighbor of b_1 and c_i in H, a contradiction.

Hence we may assume that d is a common neighbor of p and c_i in G. By Lemma 4, $N_{cl(G)}(p)$ is complete, and thus ad is an edge in $cl(G)$. But then b_1xpdc_i is a path in $N_{cl(G)}(a)$, so H is not permanent by Lemma 5. This finishes the proof. $\square$

Proof of Proposition 3. If G has the hourglass property, then in particular, it has the permanent hourglass property. By Proposition 6 (used once for each local completion), so does $cl(G)$. However, as $cl(cl(G)) = cl(G)$, all induced hourglasses in $cl(G)$ are permanent, so $cl(G)$ in fact has the hourglass property as required. □

4 Collapsible graphs

We utilize the concept of a collapsible graph, introduced by Catlin [3] (see also [4]). A graph G is *collapsible* if for any subset $X \subset V(G)$ of even size, one can find a connected spanning subgraph $H \subset G$ such that the set of vertices v with odd degree $d_H(v)$ is precisely X. If H is any subgraph of G, then the graph G/H is obtained by contracting H to a single vertex, discarding any loops but keeping all multiple edges.

Theorem 7 (Catlin [3]) *Let H be a collapsible subgraph of G. Then G has a spanning closed trail if and only if G/H does.*

A large supply of collapsible graphs is provided by the following theorem.

Theorem 8 (Catlin [3]) *Any 4-edge-connected graph is collapsible.*

A different class of collapsible graphs is obtained from the following remarkable result of Lai [6] (conjectured by P. Catlin as a strengthening of a conjecture due to Paulraja [8]).

Theorem 9 (Lai [6]) *Let G be a 2-connected graph with minimum degree $\delta(G) \geq 3$. If every edge of G is contained in a cycle of length at most 4, then G is collapsible.*

5 Line graphs

By Section 3, we may restrict our attention to the class of line graphs of triangle-free graphs. This offers us the advantage of passing to the preimage G of the line graph $L(G)$. We first need to interpret the hourglass condition in this setting.

An *I-tree* in G is any subgraph of G isomorphic to the (unique) tree J on 6 vertices, 2 of which have degree 3 in J. (Note that the tree is shaped like the letter I, see Fig. 3.) To describe an I-tree, we only give its edges, listing the edge joining the degree 3 vertices as the first one. A graph G has the *I-tree property* if in any I-tree $J \subset G$, there are two vertices of distance 3 in J that are adjacent in G.

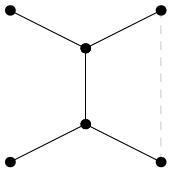

Figure 3: An I-tree (black) for which the I-tree property is satisfied.

Lemma 10 *A triangle-free graph G has the I-tree property if and only if its line graph $L(G)$ has the hourglass property.*

Proof. To prove the 'if' part, let $L(G)$ have the hourglass property and let $J = \{ xy, xx_1, xx_2, yy_1, yy_2 \}$ be an I-tree in G. The edges of J constitute an induced hourglass in $L(G)$, and so there is an edge $e \neq xy$ adjacent to, say, both xx_1 and yy_1. Since G is triangle-free, $e \notin \{ xy_1, yx_1 \}$, and so $e = x_1y_1$. Since J was arbitrary, we have established the I-tree property for G. The 'only if' implication is even more straightforward. □

The characterization of the preimages of hamiltonian line graphs is well-known. Recall that a closed trail T in a graph G is *dominating* if $G - V(T)$ is an edgeless graph.

Theorem 11 (Harary–Nash-Williams [5]) *The line graph $L(G)$ of a graph G is hamiltonian if and only if G has a dominating closed trail.*

A graph is *essentially k-edge-connected* if every edge cut of size less than k is trivial (no more than one component contains any edges). It is easy to see that G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected.

We shall derive Theorem 1 directly from the following proposition. One definition: to *suppress* a degree 2 vertex means to contract one of the edges incident with it (discarding the loop).

Proposition 12 *Any essentially 4-edge-connected, triangle-free graph with the I-tree property has a dominating closed trail.*

Proof. Let G be a graph with the stated properties. Let A be the set of vertices of degree 1 in G. The graph $G - A$ has no degree one vertices, for otherwise we could find an essential 1-cut in G. Similarly, every vertex of degree 2 in $G - A$ has degree 2 in G.

Let $B_\diamond \subset V(G - A)$ be the set of vertices of degree 2 contained in some 4-cycle, and denote the set of all other degree 2 vertices of $G - A$ by B. (The proof is illustrated in Fig. 4.) Define $G_\diamond$ to be the graph obtained from $G - A$ by suppressing all vertices in $B_\diamond$. We aim to show that each component of $G_\diamond - B$ is collapsible.

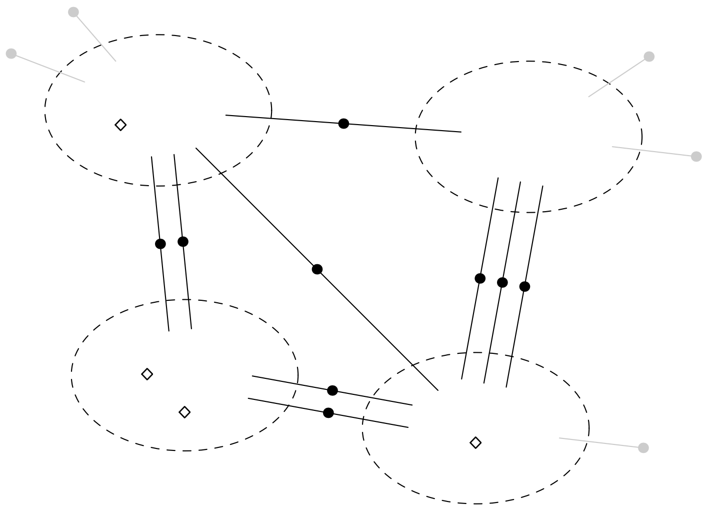

Figure 4: The structure of the graph G. Vertices in A are represented by grey dots, those in B by black ones, and those in $B_\diamond$ by $\diamond$. The ovals correspond to the components of $G_\diamond - B$.

Thus let C be a component of $G_\diamond - B$. If C is *trivial* (i.e. it consists of a single vertex), there is nothing to prove. We thus assume that C contains at least two vertices. It is our aim to verify the hypotheses of Theorem 9 for C.

We claim that the minimum degree in C is at least 3. Suppose, to the contrary, that $v \in V(C)$ is a vertex of degree less than 3. Clearly $d_{G-A}(v) \geq 3$ for otherwise it would have been either deleted or suppressed. Hence v has a neighbor $w_1 \in B$ in the graph $G - A$. We claim that in fact, it has at least two neighbors in B^-. If not, then the edges of C incident with v, together with the edge $e \in E(G - A)$ incident with w_1 but not with v, constitute an essential cut in G of size at most 3. This is impossible, so v has another neighbor $w_2 \in B$.

By our assumption that C is non-trivial, v is adjacent to a vertex $z \in V(C)$. Similarly as for v, we have $d_{G-A}(z) \geq 3$, so we may choose two neighbors $y_1, y_2 \in V(G-A)$ of z. Since G contains no triangles, $w_i \notin \{ y_1, y_2 \}$ for $i = 1, 2$. Thus the edge set $\{ vz, vw_1, vw_2, zy_1, zy_2 \}$ induces an I-tree. By the I-tree property, there is an edge between some w_i and some y_j, which yields a 4-cycle vzy_jw_i containing w_i, again a contradiction. We have shown that $\delta(C) \geq 3$. The same argument shows that every edge of C is contained in a 4-cycle of C.

Finally, we need to show that C is 2-connected. To begin with, C cannot be just a single edge, since $\delta(C) \geq 3$. Assume thus that u is a cut-vertex of C, and choose its neighbors u_1, u_2 in different components of $C - u$. Since $\delta(C) \geq 3$, u has a third neighbor u_3. Similarly, u_1 has at least two neighbors a, b besides u. If $u_3 \notin \{ a, b \}$, we may consider the I-tree with edges $\{ uu_1, u_1a, u_1b, uu_2, uu_3 \}$. Since an edge between u_2 and either of a or b is ruled out (u is a cut-vertex), we must have an edge between u_3 and a or b. To sum up, u_3 is adjacent either to u_1 or to one of its neighbors other than u. A symmetric argument shows the same for u_2 in place of u_1. But then u_3 shows that u is not a cut-vertex.

We have proved that every non-trivial component C of $G_\diamond - B$ satisfies the requirements of Theorem 9 and is therefore collapsible. Let H be the graph obtained from $G_\diamond$ by suppressing all vertices in B. Consider all non-trivial components C of $G_\diamond - B$ as subgraphs of H, contract them and suppress all vertices in B. Each vertex of the resulting graph H' corresponds to a component of $G' - B$, each edge of H' corresponds to a vertex in B. It is easy to see that H' is 4-edge-connected, for every edge cut in H' gives rise to an *essential* edge cut in G of the same size. By Theorem 8, H' has a spanning closed trail, and hence Theorem 7 implies that H has a spanning closed trail T. It is routine to check that the corresponding closed trail in G dominates every edge adjacent to a vertex in A, B or $B_\diamond$, and hence it is a dominating closed trail in G. The proof is complete. □

We now prove our main theorem.

Proof of Theorem 1. Let H be a 4-connected claw-free graph satisfying the hourglass condition. By Proposition 3 and Theorem 2, we may assume that H is a line graph, say $H = L(G)$, where G is essentially 4-connected and triangle-free. By Lemma 10, G has the I-tree property. By Proposition 12, G has a dominating closed trail, and hence, by Theorem 11, H is hamiltonian. □

Remark 13 The assumption of Theorem 1 that G is 4-connected cannot be relaxed to include 3-connected graphs, even with a lower bound on the minimum degree. This is demonstrated by the following example. Let k be an integer. Subdivide each edge of the Petersen graph by one vertex, attach k pendant edges to each vertex of degree 3, and denote the line graph of the resulting graph by H. Then H is 3-connected with minimum degree $\delta(H) = k + 2$. Since H contains no induced hourglass, it trivially has the hourglass property. It is, however, non-hamiltonian.

References

[1] H. J. Broersma, M. Kriesell and Z. Ryjáček, "On factors of 4-connected claw-free graphs", *J. Graph Theory* **37** (2) (2001) 125–136.

[2] J. Brousek, Z. Ryjáček and I. Schiermeyer, "Forbidden subgraphs, stability and hamiltonicity", *Discrete Math.* **197/198** (1999) 29–50.

[3] P. Catlin, "A reduction method to find spanning eulerian subgraphs", *J. Graph Theory* **12** (1) (1988) 29–45.

[4] P. Catlin, "Supereulerian graphs: a survey", *J. Graph Theory* **16** (2) (1992) 177–196.

[5] F. Harary and C. St. J. A. Nash-Williams, "On eulerian and hamiltonian graphs and line graphs", *Can. Math. Bull.* **8** (1965) 701–710.

[6] H. J. Lai, "Graphs whose edges are in small cycles", *Discrete Math.* **94** (1) (1991) 11–22.

[7] M. M. Matthews and D. P. Sumner, "Hamiltonian results in $K_{1,3}$-free graphs", *J. Graph Theory* **8** (1984) 139–146.

[8] P. Paulraja, "Research problem", *Discrete Math.* **64** (1987) 109.

[9] Z. Ryjáček, "On a closure concept in claw-free graphs", *J. Comb. Theory Ser. B* **70** (2) (1997) 217–224.