The conjectures and around

A warm-up survey

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- Perhaps there is a link to double cycle covers and nowhere zero flows, or other conjectures; there is a link to the P vs NP millennium problem.
- More details can be found in the **survey paper** published in Graphs and Combinatorics in 2012.

The first two conjectures

Matthews & Sumner, 1984:

Conjecture (MS-Conjecture)

Every 4-connected **claw-free graph** *is* **hamiltonian**.

Thomassen, 1986:

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We start with some **terminology** needed to understand the above statements and their relationship.

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- If H is a graph, then the **line graph of** H, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two distinct vertices in $L(H)$ are adjacent if and only if their corresponding edges in H share an end vertex (with a straightforward extension in case of multigraphs).

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- Which graphs are line graphs and which are not?

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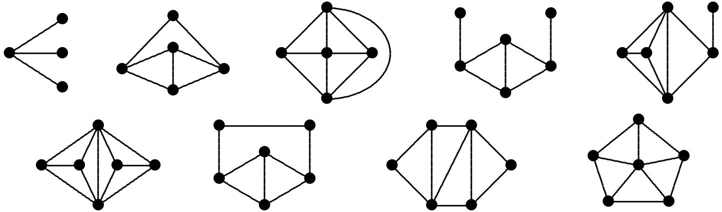

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- Direct inspection of Beineke's result shows:

every line graph is claw-free

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Every 4-connected line graph is hamiltonian.

- Since line graphs are claw-free, the first conjecture is **stronger** than the second one.
- Or are they **equivalent**? (A question Herbert Fleischner posed during the EIDMA workshop on Hamiltonicity of 2-tough graphs, Hölterhof, Enschede, November 19-24, 1996.)

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- If $G[N(v)]$ is **connected** and not a complete graph, all edges are added to turn $G[N(v)]$ into a **complete** graph.
- This procedure is repeated in the new graph, etc., until it is impossible to add any more edges.

The two conjectures are equivalent

Theorem (ZR, 1997)

Let G be a **claw-free** *graph. Then*

(*i*) *the closure $cl(G)$ is uniquely determined,*

(*ii*) *$cl(G)$ is hamiltonian if and only if G is hamiltonian,*

(*iii*) *$cl(G)$ is the* **line graph** *of a triangle-free graph.*

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Theorem (Kaiser and Vrána, 2012)

Every **5-connected** *claw-free graph with* **minimum degree at least 6** *is hamiltonian.*

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- A *closed trail* (circuit) is a connected eulerian subgraph, i.e., a connected subgraph in which all degrees are even.
- A **dominating closed trail** (DCT or D-circuit) is a closed trail T such that every edge has at least one endvertex on T.

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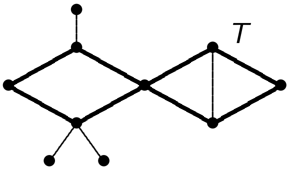

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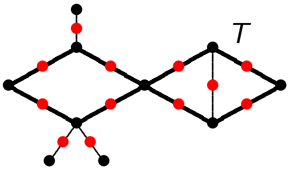

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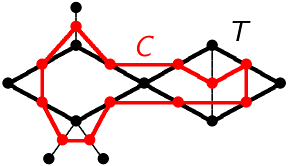

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- Hence 4-edge-connected graphs contain a **spanning** closed trail, in particular a DCT.
- So line graphs of 4-edge-connected graphs are hamiltonian (and *Hamilton-connected*).

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Conjecture (Ash & Jackson, 1984)

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Fleischner and Jackson (1989) proved that this conjecture is **equivalent** to the others.

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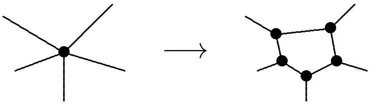
Main ingredient: *cubic inflation*



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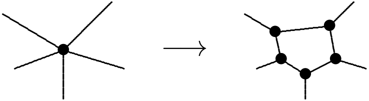

Fleischner, Jackson (1989):
Let H be an essentially 4-edge-connected graph of minimum degree $\delta(G) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Then some cubic inflation of H at v is essentially 4-edge-connected.

Two weaker (?) conjectures from the same paper

Stated in the paper of Fleischner and Jackson (1989).

Conjecture (Jaeger, ?)

Every cyclically 4-edge-connected cubic graph G has a cycle C such that $G - V(C)$ is acyclic.

Conjecture (Bondy, ?)

Every cyclically 4-edge-connected cubic graph G on n vertices has a cycle C of length at least $c \cdot n$, for some constant c with $0 < c < 1$.

It is obvious that the conjecture of Ash-Jackson implies the conjecture of Jaeger, and one can show that the conjecture of Jaeger implies the conjecture of Bondy.
Are they equivalent?

A partial relation to Bondy's conjecture

K. Ozeki, Domažlice 2013:

(a) Bondy's conjecture implies the following statement:
Every 4-connected line graph G has a cycle C of length at least $c \cdot |V(G)|$, for some constant c with $0 < c < 1$.

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(a) Bondy's conjecture implies the following statement:
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(b) The following statements are equivalent:
  (i) *Every 4-connected line graph G with $\delta(G) \geq 5$ is hamiltonian.*
  (ii) *Every 4-connected line graph G with $\delta(G) \geq 5$ has a cycle C of length at least $c \cdot |V(G)|$, for some constant c with $0 < c < 1$.*

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Kochol (2000) proved that it is **equivalent** to the others.
One direction is obvious. For the other direction, he was assuming a counterexample to the previous conjecture and used it as a black box building block. In combination with an auxiliary gadget that is almost cubic and not 3-edge-colorable he constructed a counterexample to the F-Conjecture.

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From non-3-edge colorable cubic graphs to snarks

A *snark* is a cyclically 4-edge-connected cubic graph of **girth at least** 5 that is not 3-edge-colorable.

Conjecture (Snark-Conjecture)

Every snark has a DC.

The above conjecture is also **equivalent** to the others, as shown by Broersma, Fijavž, Kaiser, Kužel, ZR & Vrána (2008).
To date this is the **seemingly weakest** conjecture equivalent to the others.
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Let us turn to some **seemingly stronger** conjectures.

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The equivalence was extended by Fleischner & Kochol (2002) by requiring a DC through any two given edges.

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Technique: A-contractible graphs

Let F be a graph and $A \subset V(F)$. Then F is *A-contractible*, if for every even subset $X \subset A$ and for every partition $\mathcal{A}$ of X into two-element subsets, the graph $F^{\mathcal{A}}$ has a DCT containing all vertices of A and all edges of $E(\mathcal{A})$.

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ZR, Schelp 2003

(*i*) *A connected graph F is A-contractible if and only if, for any H such that $F \subset H$ and $A_H(F) = A$, H has a DCT if and only if $H|_F$ has a DCT.*

(*ii*) *If F is collapsible, then F is A-contractible for any $A \subset F(F)$.*

(*iii*) *Every collapsible graph F is $V(F)$-contractible.*

Variations involving subgraphs

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Conjecture (Broersma, Fijavž, Kaiser, Kužel, ZR & Vrána, 2008)

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Conjecture (Kužel, 2008)

Any subgraph H of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ *and* $|V_2(H)| = 4$ *is* $V_2(H)$*-dominated.*

Conjecture (Kužel, Ryjáček & Vrána, 2012)

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Relation to the Nash-Williams conjecture

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H - 4-regular:
Transition system in H: a partition T of the 4 edges at every vertex into 2 sets of size 2.
H is *T-hamiltonian* if H contains a spanning closed trail which follows T at every vertex visited twice.

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A. Hoffmann-Ostenhof 2013(?):
The DCC is equivalent to the NWC.*

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ZR & Vrána (2011) extended the equivalence to claw-free graphs.

Conjecture

Every 4-connected claw-free graph is Hamilton-connected.

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Extended to claw-free graphs (ZR, Vrána, 2014)

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If yes, it implies that
Thomassen's Conjecture cannot be true, unless P=NP.

2-factors

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If we drop the connectivity condition of the 2-regular spanning subgraph, we move from a hamiltonian cycle to a *2-factor*.

Enomoto, Jackson, Katerinis & Saito (1985): *every* 2-tough *graph has a 2-factor.*

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It does not seem easy to use this as a starting point to show that there is a 2-factor with only one component, although there are some results that give upper bounds on the number of components. These results are beyond the scope of this talk.

Relaxing the 4-connectedness and adding something else

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Question: how far can we decrease the 9 by raising the 3 to 4 in the theorem?

Restrictions on the class of graphs

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Lai (1994) proved the following partial affirmative answer to Thomassen's Conjecture.

Theorem

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Theorem

Let G be a graph such that $L(G)$ is 4-connected and every vertex of degree 3 in G is on an edge of multiplicity at least 2 or on a triangle of G. Then $L(G)$ is Hamilton-connected.

Restrictions on the class of graphs

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For **quasi claw-free** graphs, i.e., in which all vertices u, v at distance 2 have a common neighbor w with $N(w) \subseteq N[u] \cup N[v]$, we have:

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A **common approach** is the following.

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There are many results along these lines.

A **common approach** is the following.
Take the preimage, delete degree 1 vertices and suppress degree 2 vertices, then try to show that the resulting graph (called the *core of G*) has a suitable spanning (closed) trail. Tools: two edge-disjoint spanning trees, or collapsibility, or advanced closure concepts.

Restrictions on the class of graphs

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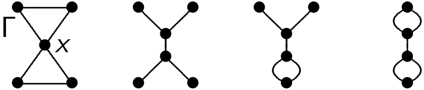
The *hourglass*: the unique graph Γ with degree sequence $4, 2, 2, 2, 2$.

Γ is a line graph and, in multigraphs, it has three nonisomorphic preimages:



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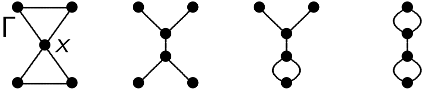

Theorem (Folklore, 90's)

Every 4-connected (claw, hourglass)-free graph is hamiltonian.

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Theorem (Kaiser, ZR, Vrána, 2014)

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Note than an analogous result is known to be true in planar graphs (a consequence of a 1997 result by Sanders).

Thank you