Tutte trail on plane graph

Adthasit Sinna Bill Jackson

School of Mathematical Science
Queen Mary, University of London

6 February 2015

Outline

1. Introduction
   - Definition
   - Hamiltonian Problem

2. Main Result
   - Main Theorem
   - Sketch of Proof

Outline

1. Introduction
   - Definition
   - Hamiltonian Problem
2. Main Result
   - Main Theorem
   - Sketch of Proof

Definition

- A graph $G = (V(G), E(G))$ where $V(G)$ is vertex set and $E(G)$ is an edge set.
- A **trail (closed trail)** is a walk in which all the edges are distinct.
- A **path (cycle)** is a walk in which all the vertices are distinct.
- A **Hamilton path(cycle)** is a path(cycle) which uses all vertices exactly once.
- A graph that contains a Hamilton cycle is a **Hamiltonian graph**.

Definition

- A graph is **k-connected** if any subgraph formed by removing any $k - 1$ vertices is still connected.
- A graph is **k-edge-connected** if any subgraph formed by removing any $k - 1$ edges is still connected.
- A **component** is a maximal connected subgraph.
- A **block** is either a maximal 2-connected subgraph, or an isolated vertex.
- A **edge-block** is either a maximal 2-edge-connected subgraph, or an isolated vertices.

Definition

Definition

Let H be a subgraph of G, the H-bridges are defined as follow.
(i) A trivial H-bridge in G is an edge in $E(G)\backslash E(H)$ with both ends in $V(H)$.
(ii) A non-trivial H-bridge in G is a component K of $G\backslash H$ together with all vertices of H adjacent to vertices of K and all edges with one end in H and the other in K.
Moreover, the vertices of attachment of a H-bridge B in G are $V(B) \cap V(H)$.

Outline

1 Introduction
- Definition
- Hamiltonian Problem

2 Main Result
- Main Theorem
- Sketch of Proof

Hamiltonian Problem

Definition

Let G be a graph and F be a subgraph of G. An *F-Tutte path(cycle)* of G is a path(cycle) P of G such that
(i) Each P-bridge of G has at most three vertices of attachment and
(ii) Each P-bridge of G containing an edge of F has at most two vertices of attachment.

- Tutte (1956) show that "2-connected plane graph has a F_G-Tutte cycle."
  where F_G is an outer walk of G.

Hamiltonian Problem

Definition

Let G be a graph and F be a subgraph of G. An *F-Tutte path(cycle)* of G is a path(cycle) P of G such that
(i) Each P-bridge of G has at most three vertices of attachment and
(ii) Each P-bridge of G containing an edge of F has at most two vertices of attachment.

- Tutte (1956) show that "2-connected plane graph has a F_G-Tutte cycle."
  where F_G is an outer walk of G.
- Implies "4-connected plane graph has Hamilton cycle"

Hamiltonian Problem

Definition

Let G be a graph and F be a subgraph of G. An *F-Tutte path(cycle)* of G is a path(cycle) P of G such that
(i) Each P-bridge of G has at most three vertices of attachment and
(ii) Each P-bridge of G containing an edge of F has at most two vertices of attachment.

- Tutte (1956) show that "2-connected plane graph has a F_G-Tutte cycle."
  where F_G is an outer walk of G.
- Implies "4-connected plane graph has Hamilton cycle"
- Tutte graph, 3-connected 3-regular plane graph, is non-hamiltonian.

Hamiltonian Problem

- Thomassen (1983) and Sander (1997) show that "2-connected plane graph has a F_G-Tutte path from u to v containing e for any $u, v \in V(G)$ and $e \in E(F_G)$"

Hamiltonian Problem

- Thomassen (1983) and Sander (1997) show that "2-connected plane graph has a F_G-Tutte path from u to v containing e for any $u, v \in V(G)$ and $e \in E(F_G)$"
- Implies Plummer's Conjecture : "4-connected plane graph is Hamiltonian-connected" (a Hamiltonian path connecting any two prescribed vertices.)

Conjecture

- Every 4-connected claw-free graph is hamiltonian.
  (Mathews-Summer, 1984)

Conjecture

- Every 4-connected claw-free graph is hamiltonian. (Mathews-Summer, 1984)
- Every 4-connected line graph is hamiltonian.(Thomassen, 1986)

Conjecture

- Every 4-connected claw-free graph is hamiltonian. (Mathews-Summer, 1984)
- Every 4-connected line graph is hamiltonian.(Thomassen, 1986)
- Every essentially 4-edge-connected graph contains a dominating closed trail.

Conjecture

- Every 4-connected claw-free graph is hamiltonian. (Mathews-Summer, 1984)
- Every 4-connected line graph is hamiltonian.(Thomassen, 1986)
- Every essentially 4-edge-connected graph contains a dominating closed trail.
- Every essentially 4-edge-connected 3-regular graph has a dominating cycle. (Ash and Jackson)

Conjecture

- Every 4-connected claw-free graph is hamiltonian. (Mathews-Summer, 1984)
- Every 4-connected line graph is hamiltonian.(Thomassen, 1986)
- Every essentially 4-edge-connected graph contains a dominating closed trail.
- Every essentially 4-edge-connected 3-regular graph has a dominating cycle. (Ash and Jackson)
- Every essentially 4-edge-connected 3-regular graph that is not 3-edge-colorable has a dominating cycle. (Fleischner)

Conjecture

- Every 4-connected claw-free graph is hamiltonian. (Mathews-Summer, 1984)
- Every 4-connected line graph is hamiltonian.(Thomassen, 1986)
- Every essentially 4-edge-connected graph contains a dominating closed trail.
- Every essentially 4-edge-connected 3-regular graph has a dominating cycle. (Ash and Jackson)
- Every essentially 4-edge-connected 3-regular graph that is not 3-edge-colorable has a dominating cycle. (Fleischner)
- are all equivalent.

Outline

1. Introduction
   - Definition
   - Hamiltonian Problem
2. Main Result
   - Main Theorem
   - Sketch of Proof

Main Theorem

Definition

Let G be a graph and F be a subgraph of G. An *F-Tutte trail(closed trail)* of G is a trail(closed trail) H of G such that
(i) Each component of $G \backslash V(H)$ has at most three edges connecting it to H.
(ii) Each component of $G \backslash V(H)$ containing a vertex of F has at most two edges connecting it to H.

- For any plane graph G, the outer walk of G is denoted by F_G.

Main Theorem

Definition

Let G be a graph and F be a subgraph of G. An *F-Tutte trail(closed trail)* of G is a trail(closed trail) H of G such that
(i) Each component of $G \backslash V(H)$ has at most three edges connecting it to H.
(ii) Each component of $G \backslash V(H)$ containing a vertex of F has at most two edges connecting it to H.

- For any plane graph G, the outer walk of G is denoted by F_G.
- A seperation (G_1, G_2) of G is a *k-seperation* if $|V(K_1)|, |V(K_2)| \geq k + 1$ and $|V(G_1) \cap V(G_2)| = k$.

Main Theorem

Theorem

Let G be a 2-edge-connected plane graph.
(a) If $u, v \in V(G)$ and $e \in E(F_G)$, then there is an F_G-Tutte trail in G from u to v containing e.
(b) If $E(F_G) \geq 3$ and $e_1, e_2, e_3 \in E(F_G)$, then there is an F_G-Tutte closed trail in G containing e_1, e_2 and e_3.
(c) If $E(F_G) = \{e_1, e_2\}$, then there is an F_G-Tutte closed trail in G containing e_1 and e_2.

Outline

1 Introduction
- Definition
- Hamiltonian Problem

2 Main Result
- Main Theorem
- Sketch of Proof

Sketch of Proof (a)

- We proceed by contradiction. Suppose the theorem is false and choose a counterexample G such that $|V(G)|$ is small as possible and, subject to this condition, $|E(G)|$ is as small as possible. It can checked that the theorem is true when $|V(G)| \leq 3$. So we have $|V(G)|| \geq 4$.

Sketch of Proof (a)

- We proceed by contradiction. Suppose the theorem is false and choose a counterexample G such that $|V(G)|$ is small as possible and, subject to this condition, $|E(G)|$ is as small as possible. It can checked that the theorem is true when $|V(G)| \leq 3$. So we have $|V(G)|| \geq 4$.
- **Case 1**: G has a 1-seperation.
- **Case 2**: $v \in V(F_G)$ and G has a 2-seperation which seperate v and e.
- **Case 3**: $v \in V(F_G)$ and G has no 2-seperation in Case 2.
- **Case 4**: $u, v \notin V(F_G)$ and G has no 2-seperation in Case 2.

Case 1: G has 1-seperation.

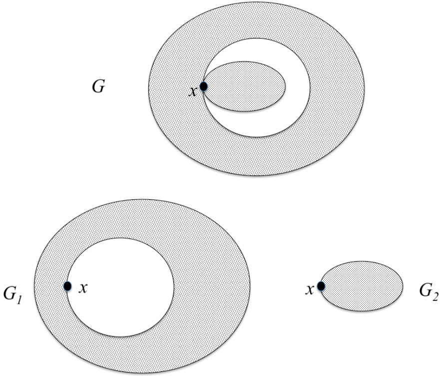

Figure: the graphs G, G_1 and G_2 when $x \notin V(F_G)$.

Case 1: G has 1-seperation.

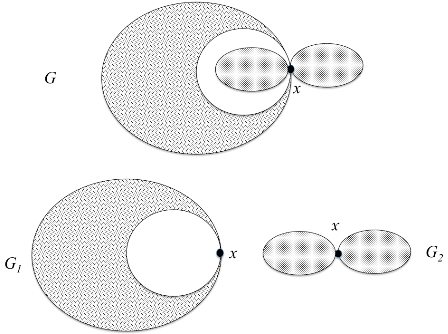

Figure: the graphs G, G_1 and G_2 when $x \in V(F_G)$.

Case 2: $v \in V(F_G)$ and G has 2-seperation.

- $v \in V(G_1)$, $e \in E(G_2)$ and $V(G_1) \backslash \{v, x, y\} \neq \emptyset$

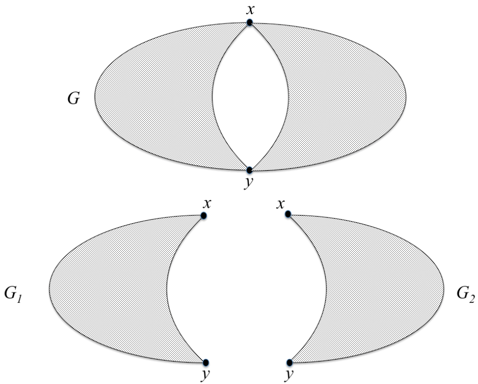

Figure: the graphs G, G_1, G_2.

Case 2: $v \in V(F_G)$ and G has 2-seperation.

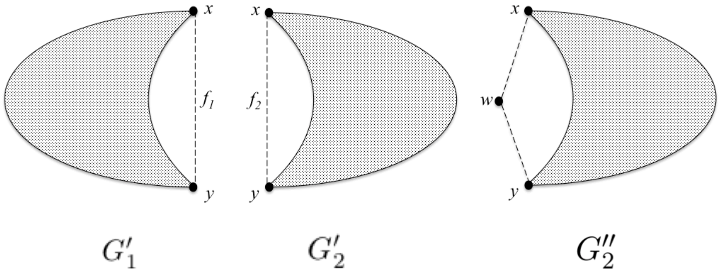

Figure: the graphs G_1', G_2' and G_2''.

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

- Let $F \subseteq F_G$ and $H \subseteq G \backslash V(F)$.
- An $(F \cup H)$-*bridge group* A is the union of a maximal $(F \cup H)$-bridge X together with all $(F \cup H)$-bridges Y such that $|V(Q_Y)| \geq 2$ and $Q_Y \subset Q_X$. We put $Q_A = Q_X, p_A = p_X$ and $q_A = q_X$ where X is the maximal $(F \cup H)$-bridge in A.
- An (F, H)-*connector* in G is a bridge of $F \cup H$ in G with its vertices of attachment in both F and H.
- An (F, H)-*connector group* L is an $(F \cup H)$-bridge group which contains an (F, H)-connector in G.

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

- Let P_1 be a segment of F_G from v to an end vertex of e such that $e \notin E(P_1)$ and $u \notin P_1 \backslash \{v\}$ ($P_1 = v$ if v is an end vertex of e).
- w be the adjacent vertex of v on F_G which is not in P_1.
- v_1 be the end of e which is not on P_1
- $P_2 = F_G \backslash V(P_1)$
- $H = G \backslash V(P_1)$

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

- There is a unique block B of H containing P_2.

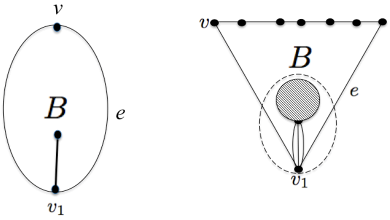

Figure: The block B in the case v_1 has one neighbor in H.

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

- we define B' as the edge-block of H containing B when $B \neq K_2$ (possibly $B' = B$). If $B = K_2$, we define $B' = \{v_1\}$.

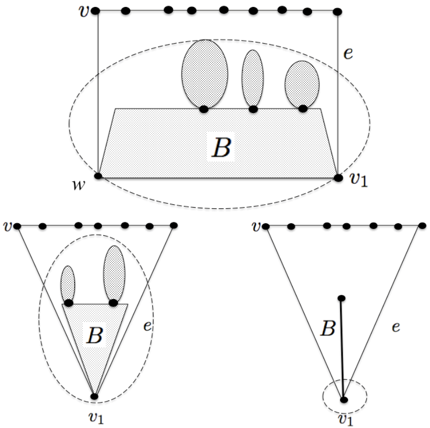

Figure: The structure of B'.

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

- Choose an $F_{B'}$-Tutte trail T' from v_1 to u' containing an edge of $F_{B'}$ incident to w in B'
- Let $T = P_1 \cup T' \cup \{e\}$ is a vu'-trail.
  We will modify T by diverting it into each $(P_1 \cup T')$-bridge group J

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

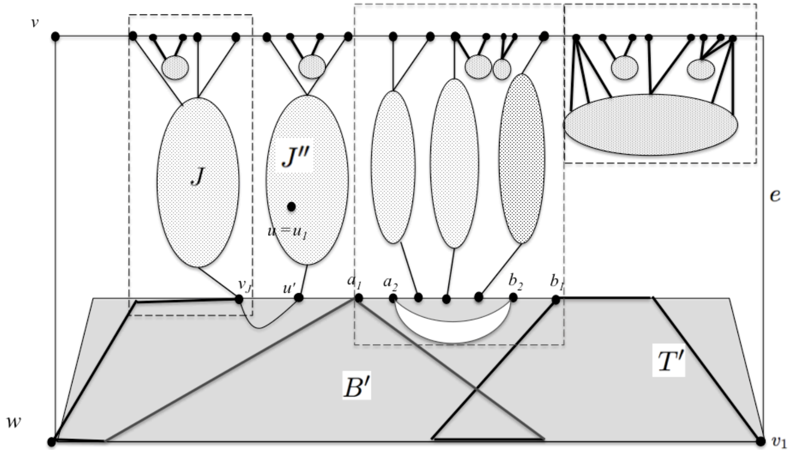

Figure: $(P_1 \cup T')$-bridge groups of G.

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

- **Subcase 3.1:** J has no edge of attachment on T'.
  Then, by induction on (a), $J \cup Q_J$ has a $p_J q_J$-Tutte trail T_J from p_J to q_J. In T, we replace Q_J by T_J for each such J.

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

- **Subcase 3.2:** J has one edge of attachment on T'.
  Let v_J be the vertex of attachment of J in T' and J^* be the union of J, Q_J and the new edge $e_J = p_J v_J$(See Fig. 8). Then J^* is 2-edge-connected and by induction on (a), has an F_{J^*}-Tutte trail T_J from q_J to v_J containing e_J. In T, we replace Q_J by $T_J - v_J$ for each such J.

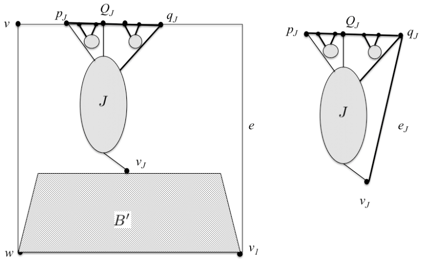

Figure: The structure of J^*.

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

Subcase 3.3: J has two edges of attachment on T'.
Let $J_1 = (J \cup Q_J) \backslash \{a_1, b_1\}$ and J_2 be the edge-block of J_1 containing Q_J.

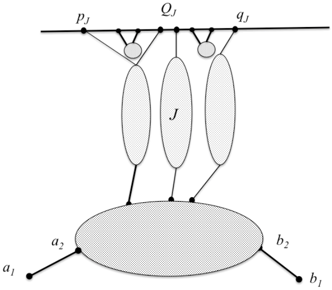

Figure: The structure of J in Subcase 3.3.

Case 3: $v \in V(F_G)$ and G has no 2-seperation in Case 2

Subcase 3.3.1: $J_1 = J_2$.
Then, by induction on (a), J_2 has an F_{J_2}-Tutte trail T_1 from p_J to q_J containing an edge of F_{J_2} incident to a_2. In T, we replace Q_J by T_1. Note that if $b_2 \notin T_1$, the component S of $J_1 - T_1$ containing b_2 has exactly two edges connecting it to T_1 and then has exactly three edges connecting it to T.

Subcase 3.3.2: $J_1 \neq J_2$.

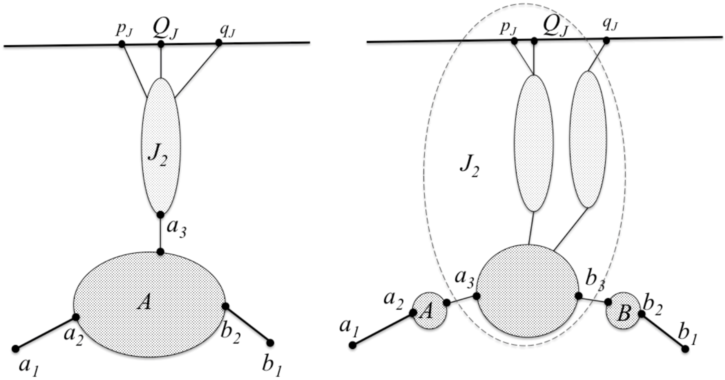

Figure: The structure of J when $J_1 \neq J_2$.

- If $u' = u_1 = u$, then T is the desired Tutte trail in G.
- If $u = v(u' = u_1 = w)$, then $T \cup \{vw\}$ is the desired Tutte trail.
- If $u \neq u'$, then $u \in V(J'')$. Let K be the graph obtained from the union of $J'', Q_{J''}$ and the new edge $e_K = q_J u'$. Then K is 2-edge-connected and, by induction on (a), has F_K-Tutte trail T_K from p_J to u containing e_K. Hence $(T - Q_{J''}) \cup (T_K - e_K)$ is a desired F_G-Tutte trail from u to v containing e.

Case 4: $u, v \notin V(F_G)$ and G has no 2-seperation in Case 2.

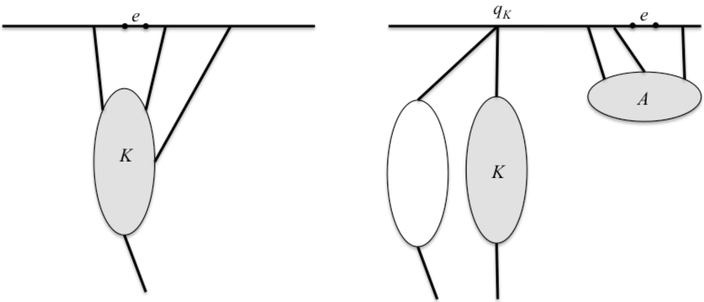

Figure: The structure of K.

Case 4: $u, v \notin V(F_G)$ and G has no 2-seperation in Case 2.

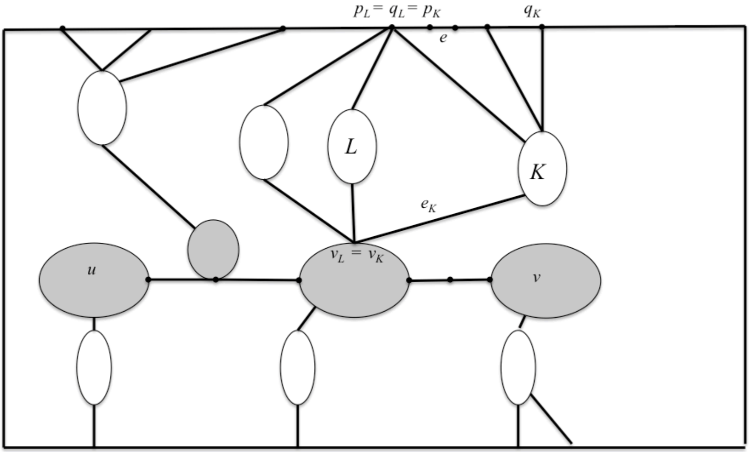

Figure: The structure of K and L when $p_K = q_L$.

Case 4: $u, v \notin V(F_G)$ and G has no 2-seperation in Case 2.

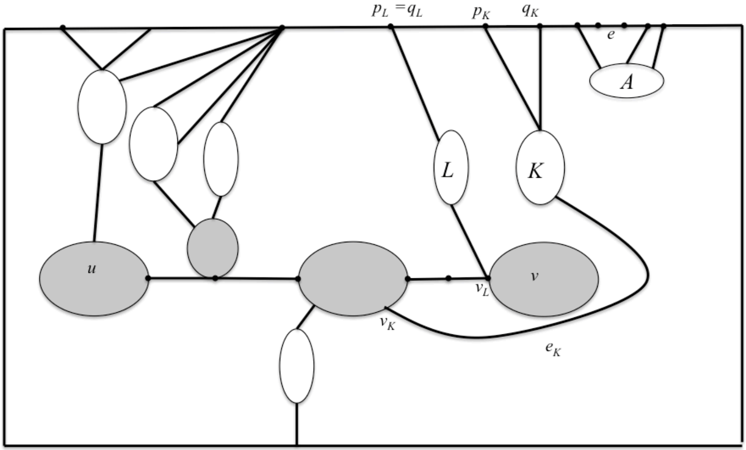

Figure: The structure of K and L when $p_K \neq q_L$.

Case 4: $u, v \notin V(F_G)$ and G has no 2-seperation in Case 2.

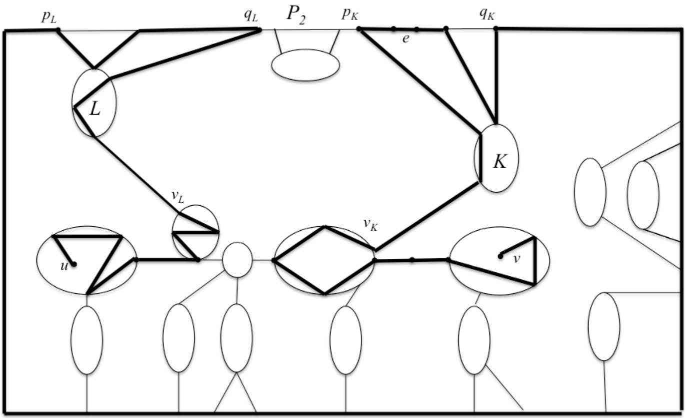

Figure: The structure of T.