6-decomposition of snarks

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Vancouver, January 2012

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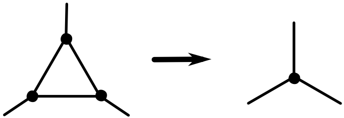
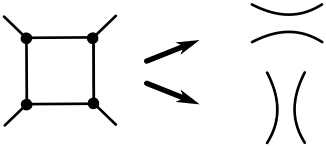
Snark reductions

(a) Reduction of a dipole



(b) Reduction of a triangle



(c) Reduction of a quadrilateral

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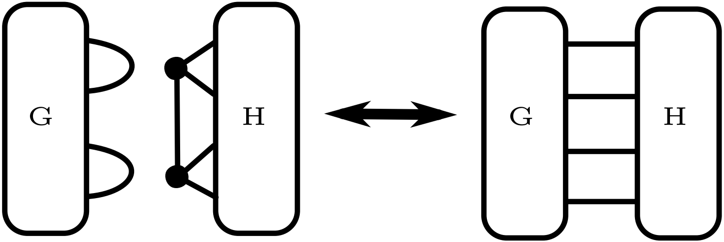
- **Example** A 5-composition of of two J_5 gives a double-star snark.

Isaac's dot product



4-decompositions of irreducible snarks

A recent nontrivial result by Chladny, Škoviera:
Irreducible snarks have a unique 4-decomposition into irreducible and indecomposable factors.

Note: An irreducible snark indecomposable along 4-cuts is 5-cyclically connected

General k-Decomposition Theorem

Theorem.[N.+ Škoviera] There is a function $f(k)$ such that for every snark $G = M \cdot N$ which is a k-composition of k-poles M, N

- either one of M, N is not colorable, (k-reduction)
- or both M, N are colorable and by adding at most $f(k)$ vertices both M, N can be completed to snarks. (k-decomposition)

PROBLEM. Min. $f(k) =?$, known $f(2) = 0$, $f(3) = 1$, $f(4) = 2$ and $f(5) = 5$,
$\mathbf{f(6)} =?$, $f(k)$ for $k \geq 7$ maybe not needed

Conjecture (Jaeger): No 7-cyclically connected snark does exist.

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- the only known prime snark is the Petersen graph

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- The colour set is the set of all colour types induced by all 3-edge-colourings of an (ordered) k-pole,
- Each colour type satisfies the Parity Lemma, The numbers of admissible colour types for $k = 2, 3, 4, 5, 6$: $1, 1, 4, 10, 31$.

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The complements for $k \leq 5$

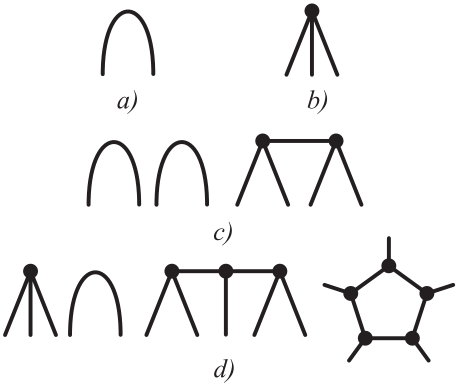

Figure: Poles representing minimal Kempe-closed sets in cases $k = 2, 3, 4, 5$.

Decomposition into black boxes

Coloring: a 3-partition $A \cup B \cup C$ of the index set $\{1, 2, \ldots, k\}$, one or two of the partition sets may be empty,

Col set: a set of admissible colorings, Parity lemma: $6 = 0 + 0 + 6, 2 + 4, 2 + 2 + 2$

Col sets of degree 6: Col set of degree 6 can be viewed as a binary vector of length 31 for $k = 6$,

Action of S_k on sets of 3-partitions

$$\psi\{a_1, a_2, \ldots, a_m\} = \{\psi(a_1), \psi(a_2), \ldots, \psi(a_m)\}$$

quasi-order: $\mathcal{A} \preceq \mathcal{B}$ if $\exists\ \psi \in S_k$ such that $\psi(\mathcal{A}) \subseteq \mathcal{B}$,

complementability: $\exists\ \psi \in S_k$ such that $\psi(\mathcal{A}) \cap \mathcal{B} = \emptyset$

Adjacent Colorings, Kempe Closed col sets

adjacent colorings: Two 3-partitions $\cup X_i$ and $\cup X_i'$ are adjacent if $\exists$ ψ of $\{1,2,3\}$ such that $X_i = X'_{\psi(i)}$ for some i and the symmetric difference $|X_r \div X'_{\psi(r)}| = 2$, if $r \neq i$.

Kempe-closed set: Some subsets of the universal set (universal graph on the 31 parity-admissible 3-partitions) are closed on Kempe switches,

Any realisable colour set is Kempe-closed.

Problem 1. Is every Kempe-closed colour set of degree 6 realisable?

Structure of the adjacency graph

vertex-set $V = \{111111\} \cup V(2+4) \cup V(2+2+2)$

edges the vertex 111111 is adjacent to all 15 vertices in $V(2+4)$

edges the induced graphs on $V(2+4)$ and $V(2+2+2)$ are vertex-transitive graphs of degrees 8 and 6, respectively,

edges the edges between $V(2+4)$ and $V(2+2+2)$ form the Tutte's 8-cage - the smallest 5-arc-transitive cubic graph.

Minimal colour open complements for $k = 6$

Solution in terms of BLACK BOXES:

- **Theorem. There is a unique family of minimal complements containing** 38 **colour sets forming complements of all maximal colour-open colsets.**

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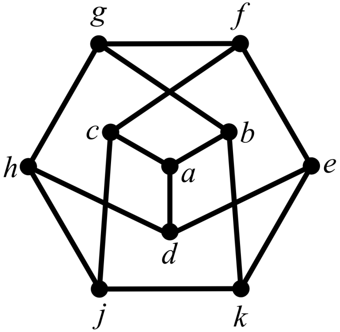
- Bad Luck: **We can realize only nine of them!!!**
- Further reduction of the realizability problem to 14 atomic colour sets is done.

Realization problem

Problem: Given Kempe closed set A of degree k, is there a k-pole P such that $Col(P) = A$?

For $k \leq 5$ the answer is positive, for $k = 6$ we have found a realization of about 2600 colour open sets, about 60 percent of all. Unfortunately we cannot do it for the remaining minimal complements from the theorem.

Petersen-like Colset, Is it realizable?



a: **16|25|34**
b: **16|24|35**
c: **14|25|36**
d: **15|26|34**
e: **15|23|46**
f: **14|23|56**
g: **13|24|56**
h: **13|26|45**
j: **12|36|45**
k: **12|35|46**

Decomposition theorem for small snarks

Theorem (Vrtak 1997). Colset of any 6-pole P in a snark with at most 30 vertices contains at least one of $Col(V_i)$, $i = 1, 2, \ldots, 14$.

Irreducible 6-poles of small order

Irreducibility: there is no 6-pole Q such that $Col(Q) \subset Col(P)$ and P is a realization of minimum order.

Proposition. There are exactly 17 irreducible 6-poles with at most 12 vertices. 14 of them are the Vrtak poles, two additional are colour closed, one is colour open, but a realization of the complement is not known.

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- **Theorem. If Vrtak's set is not complete then $f(6) \geq 20$.**

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