On dominating even subgraphs in cubic graphs

Kiyoshi Yoshimoto

Department of Mathematics
Nihon University

Pilzen, Czech
31 March 2015

joint work with R. Cada, S. Chiba, K. Ozeki
Theorem. (Fujisawa, Xiong, Y, Zhang 2007)

The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq 3n - 2$ components.

Conjecture. (Fujisawa, Xiong, Y, Zhang 2007)

The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq (2\delta - 3)n^2(\delta^2 - \delta - 1)$ components $< n\delta$ components.
2004

**Theorem.** (Fujisawa, Xiong, Y, Zhang 2007)

- The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq \frac{3n - 2}{8}$ components.
Theorem. (Fujisawa, Xiong, Y, Zhang 2007)

The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq \frac{3n - 2}{8}$ components.

Conjecture. (Fujisawa, Xiong, Y, Zhang 2007)

The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq \frac{(2\delta - 3)n}{2(\delta^2 - \delta - 1)}$ components $< \frac{n}{\delta}$ components.
Conjecture. (Fujisawa, Xiong, Y, Zhang 2007)

- The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq \frac{(2\delta - 3)n}{2(\delta^2 - \delta - 1)}$ components $< \frac{n}{\delta}$ components.

Theorem. (Jackson, Y 2007)

(Case of $\delta = 3$.) The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq 3n - 4$ components.
Conjecture. (Fujisawa, Xiong, Y, Zhang 2007)

- The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq \frac{(2\delta - 3)n}{2(\delta^2 - \delta - 1)}$ components $< \frac{n}{\delta}$ components.

- (Case of $\delta = 3$)
  The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq \frac{3n}{10}$ components.
Conjecture. (Fujisawa, Xiong, Y, Zhang 2007)

- The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq \frac{(2\delta - 3)n}{2(\delta^2 - \delta - 1)}$ components $< \frac{n}{\delta}$ components.

- (Case of $\delta = 3$)

The line graph of a graph with $\delta \geq 3$ has a 2-factor with $\leq \frac{3n}{10}$ components.

2005

Theorem. (Jackson, Y 2007)
Conjecture. (Fujisawa, Xiong, Y, Zhang 2007)

- The line graph of a graph with $\delta \geq 3$ has a 2-factor with
  \[ \leq \frac{(2\delta - 3)n}{2(\delta^2 - \delta - 1)} \]
  components.
  \[ < \frac{n}{\delta} \]
  components.

(Case of $\delta = 3$)
The line graph of a graph with $\delta \geq 3$ has a 2-factor with
\[ \leq \frac{3n}{10} \]
components.

2005

Theorem. (Jackson, Y 2007)
(Case of $\delta = 3$.) The line graph of a graph with $\delta \geq 3$ has
a 2-factor with
\[ \leq \frac{3n - 4}{10} \]
components.
2005

- A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor with $\frac{n+1}{4}$ components. \hfill (Jackson, Y 2007)

- A 3-connected claw-free graph with $\delta \geq 4$ has a 2-factor with $\frac{2n}{15}$ components. \hfill (Jackson, Y 2009)
2005

- A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor with $\frac{n + 1}{4}$ components.  
  \textit{(Jackson, Y 2007)}

- A 3-connected claw-free graph with $\delta \geq 4$ has a 2-factor with $\frac{2n}{15}$ components.  
  \textit{(Jackson, Y 2009)}

2006

- A claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\frac{n}{\delta - 1}$ components.  
  \textit{(Broersma, Paulusma, Y 2009)}
Theorem. (Faudree, Magnant, Ozeki and Y 2012)

2008

A line graph with $\delta \geq 7$ has a spanning subgraph in which every component is a clique of order $\geq 3$.

If $G$ is a line graph with $\delta \geq 7$, then for any independent set $S$, $G$ has a 2-factor such that each cycle contains $\leq 1$ vertex in $S$. 
Theorem. (Faudree, Magnant, Ozeki and Y 2012)

A line graph with $\delta \geq 7$ has a spanning subgraph in which every component is a clique of order $\geq 3$. 

2008
2008

Theorem. (Faudree, Magnant, Ozeki and Y 2012)

- A line graph with \( \delta \geq 7 \) has a spanning subgraph in which every component is a clique of order \( \geq 3 \).

- If \( G \) is a line graph with \( \delta \geq 7 \),
  for any independent set \( S \),
  \( G \) has a 2-factor such that
  each cycle contains \( \leq 1 \) vertex in \( S \).
2009

Theorem. (Kuzel, Ozeki, Y 2012)
2009

Theorem. \textit{(Kuzel, Ozeki, Y 2012)}

- If $G$ is a 3-connected claw-free graph with $\delta \geq 4$, for any maximum independent set $S$, $G$ has a 2-factor in which each cycle contains $\geq 2$ vertices in $S$. 
Theorem. (Ryjacek, Ozeki, Y 2015)

A 3-connected claw-free graph has a 2-factor with at most $2(\alpha + 1)$ components.

A 3-connected claw-free graph has a 2-factor with at most $n^{5/4}(\delta + 2)$ components.
Theorem. (Ryjacek, Ozeki, Y 2015)

A 3-connected claw-free graph has a 2-factor with at most $\frac{2}{5}(\alpha + 1)$ components.
Theorem. (Ryjacek, Ozeki, Y 2015)

- A 3-connected claw-free graph has a 2-factor with at most \( \frac{2}{5}(\alpha + 1) \) components.

- A 3-connected claw-free graph has a 2-factor with at most \( \frac{n}{\frac{5}{4}(\delta + 2)} \) components.
Theorem. (Cada, Chiba, Y 2015)

A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor in which every cycle contains $\geq \delta$ vertices.

Theorem. (Cada, Chiba 2013)

A 2-connected claw-free graph with $\delta \geq 7$ has a 2-factor in which longest cycle has length $\geq 2\delta + 4$. 
2010

Theorem. \textit{(Cada, Chiba, Y 2015)}

A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor in which every cycle contains $\geq \delta$ vertices.

2011

Theorem. \textit{(Cada, Chiba 2013)}

A 2-connected claw-free graph with $\delta \geq 7$ has a 2-factor in which longest cycle has length $\geq 2\delta + 4$. 
2010

Theorem. (Cada, Chiba, Y 2015)

A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor in which every cycle contains $\geq \delta$ vertices.

2011

Theorem. (Cada, Chiba 2013)

A 2-connected claw-free graph with $\delta \geq 7$ has a 2-factor in which longest cycle has length $\geq 2\delta + 4$. 
Known upper bounds of number of cycles of 2-factors of claw-free graphs
Known upper bounds of number of cycles of 2-factors of claw-free graphs

- A claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\frac{n}{\delta - 1}$ components. (Broersma, Paulusma, Y 2009)
Known upper bounds of number of cycles of 2-factors of claw-free graphs

- A claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\frac{n}{\delta - 1}$ components. (Broersma, Paulusma, Y 2009)

- A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\frac{n}{\delta}$ components. (Cada, Chiba, Y 2015)
Known upper bounds of number of cycles of 2-factors of claw-free graphs

- A claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\frac{n}{\delta - 1}$ components. (Broersma, Paulusma, Y 2009)

- A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor with at most $\frac{n}{\delta}$ components. (Cada, Chiba, Y 2015)

- A 3-connected claw-free graph has a 2-factor with at most $\frac{n}{\frac{3}{4}(\delta + 2)}$ components. (Ryjacek, Ozeki, Y 2015)
A 2-connected claw-free graph with $\delta \geq 4$ has a 2-factor with $\frac{n + 1}{4}$ components.  

(Jackson, Y 2007)

A 3-connected claw-free graph with $\delta \geq 4$ has a 2-factor with $\frac{2n}{15}$ components.  

(Jackson, Y 2009)
We consider the graph $G^*$ obtained from a $k$-edge-connected graph $G$ by
We consider the graph $G^*$ obtained from a $k$-edge-connected graph $G$ by

- removing all end vertices and
We consider the graph $G^*$ obtained from a $k$-edge-connected graph $G$ by

- removing all end vertices and
- suppressing all vertices of degree 2.
We consider the graph $G^*$ obtained from a $k$-edge-connected graph $G$ by

- removing all end vertices and
- suppressing all vertices of degree 2.

$L(G)$ is $k$-connected $\iff G^*$ is $k$-edge-connected $(k \leq 3)$. 

(Petersen 1891)
We consider the graph $G^*$ obtained from a $k$-edge-connected graph $G$ by

- removing all end vertices and
- suppressing all vertices of degree 2.

$L(G)$ is $k$-connected $\iff G^*$ is $k$-edge-connected $(k \leq 3)$.

- A 2-edge-connected cubic graph has a 2-factor.

(Petersen 1891)
A 2-edge-connected cubic graph has a 2-factor.

(Petersen 1891)
A 2-edge-connected cubic graph has a 2-factor. 
(Petersen 1891)

A 2-edge-connected graph with $\delta \geq 3$ has a spanning even subgraph. 
(Fleischner 1992)
A 2-edge-connected cubic graph has a 2-factor.  
(Petersen 1891)

A 2-edge-connected graph with $\delta \geq 3$ has a spanning even subgraph.  
(Fleischner 1992)

A 2-edge-connected simple graph with $\delta \geq 3$ has a spanning even subgraph in which every component has $\geq 4$ vertices.  
(Jackson, Y 2007)
■ A 2-edge-connected cubic graph has a 2-factor.  
  
  (Petersen 1891)

■ A 2-edge-connected graph with $\delta \geq 3$ has a spanning even subgraph.  
  
  (Fleischner 1992)

■ A 2-edge-connected simple graph with $\delta \geq 3$ has a spanning even subgraph in which every component has $\geq 4$ vertices.  
  
  (Jackson, Y 2007)

■ A 3-edge-connected graph has a spanning even subgraph in which every cycle contains $\geq 5$ vertices.  
  
  (Jackson, Y 2009)
A 2-edge-connected simple cubic graph with $\delta \geq 3$ has a 2-factor in which every cycle contains $\geq 4$ vertices. (Jackson, Y 2007)

A 3-edge-connected cubic graph has a 2-factor in which every cycle contains $\geq 5$ vertices. (Jackson, Y 2009)
A 2-edge-connected simple cubic graph with $\delta \geq 3$ has a 2-factor in which every cycle contains $\geq 4$ vertices. (Jackson, Y 2007)

A 3-edge-connected cubic graph has a 2-factor in which every cycle contains $\geq 5$ vertices. (Jackson, Y 2009)

There is an infinite family of essentially 4-edge-connected cubic graphs in which every 2-factor contains a 5-cycle. (Jackson, Y 2009)
Theorem. (Jackson, Y 2009)

- A 3-edge-connected cubic graph $F$ has a 2-factor such that every cycle of $F$ contains $\geq 5$ vertices.
Theorem. \hspace{1cm} (Jackson, Y 2009)

A 3-edge-connected cubic graph $F$ has a 2-factor such that every cycle of $F$ contains $\geq 5$ vertices.

Theorem. \hspace{1cm} (Kaiser, Skrekovski 2008)
Theorem. \hspace{1cm} (Jackson, Y 2009)

- A 3-edge-connected cubic graph $F$ has a 2-factor such that every cycle of $F$ contains $\geq 5$ vertices.

Theorem. \hspace{1cm} (Kaiser, Skrekovski 2008)

- Every graph has an even subgraph which intersects all 3-cuts and 4-cuts.
Theorem. (Jackson, Y 2009)

- A 3-edge-connected cubic graph $F$ has a 2-factor such that every cycle of $F$ contains $\geq 5$ vertices.

Theorem. (Kaiser, Skrekovski 2008)

- Every graph has an even subgraph which intersects all 3-cuts and 4-cuts.
- A 2-edge-connected cubic graph has a 2-factor which intersects all 3-cuts and 4-cuts.
Theorem. (Jackson, Y 2009)

- A 3-edge-connected cubic graph $F$ has a 2-factor such that every cycle of $F$ contains $\geq 5$ vertices.

Theorem. (Kaiser, Skrekovski 2008)

- Every graph has an even subgraph which intersects all 3-cuts and 4-cuts.
- A 2-edge-connected cubic graph has a 2-factor which intersects all 3-cuts and 4-cuts.
- A 3-edge-connected cubic graph has a 2-factor $F$ such that every cycle of $F$ contains $\geq 5$ vertices and $F$ intersects all 3-cuts and 4-cuts.
Theorem. (Kaiser, Skrekovski 2008)

A 3-edge-connected cubic graph has a 2-factor $F$ such that every cycle of $F$ contains $\geq 5$ vertices and $F$ intersects all 3-cuts and 4-cuts.

Theorem. (Cada, Chiba, Ozeki, Y 2015+)

A 3-edge-connected cubic graph has a dominating even subgraph $F$ such that each cycle of $F$ contains $\geq 6$ vertices and $F$ intersects all 3-cuts.
Theorem. (Kaiser, Skrekovski 2008)

- A 3-edge-connected cubic graph has a 2-factor $F$ such that every cycle of $F$ contains $\geq 5$ vertices and $F$ intersects all 3-cuts and 4-cuts.

Theorem. (Cada, Chiba, Ozeki, Y 2015+)

- A 3-edge-connected cubic graph has a dominating even subgraph $F$ such that each cycle of $F$ contains $\geq 6$ vertices and $F$ intersects all 3-cuts.
Sketch of the Proof.

Let $G$ be a 3-edge-connected cubic graph.
**Sketch of the Proof.**

Let $G$ be a 3-edge-connected cubic graph.

If $G$ has no 5-cycle,
Sketch of the Proof.

Let $G$ be a 3-edge-connected cubic graph.

If $G$ has no 5-cycle, 

$\implies G$ has a desired 2-factor

by the theorem of Kaiser and Skrekovski.
Sketch of the Proof.

Let $G$ be a 3-edge-connected cubic graph.

If $G$ has no 5-cycle,

$\implies G$ has a desired 2-factor

by the theorem of Kaiser and Skrekovski.

A 5-cycle $C$ of a cubic graph $G$ is called good

if there is a 3-cut $T$ such that $|\partial(C) \cap T| \geq 2$. 
Sketch of the Proof.

Let $G$ be a 3-edge-connected cubic graph.

If $G$ has no 5-cycle,

$\implies G$ has a desired 2-factor

by the theorem of Kaiser and Skrekovski.

A 5-cycle $C$ of a cubic graph $G$ is called good

if there is a 3-cut $T$ such that $|\partial(C) \cap T| \geq 2$.

If $G$ has no bad 5-cycle,

$\implies G$ has a desired 2-factor

by the theorem of Kaiser and Skrekovski.
A 5-cycle $C$ of a cubic graph $G$ is called good if there is a 3-cut $T$ such that $|\partial(C) \cap T| \geq 2$.

If $G$ has no bad 5-cycle,

$\implies G$ has a desired 2-factor by the theorem of Kaiser and Skrekovski.
Suppose $G$ has bad 5-cycles.
Suppose $G$ has bad 5-cycles.

We will reduce following subgraphs obtained from 5-cycles.

(a) $u_1$ $u_2$ $u_3$

(b) $u_1$ $u_2$ $u_3$

(c) $u_1$ $u_2$ $u_3$
Suppose $G$ has bad 5-cycles.

We will reduce following subgraphs obtained from 5-cycles.

An $i$-cell $D$ is called good if $D$ contains a good 5-cycle.
1st. We will reduce bad 2-cells $D$ recursively as follows:
1st. We will reduce bad 2-cells $D$ recursively as follows:
1st. We will reduce bad 2-cells $D$ recursively as follows:

We choose a next 2-cell which is bad in $G|D$ and contains no reduced vertices.
1st. We will reduce bad 2-cells $D$ recursively as follows:

We choose a next 2-cell which is bad in $G|D$ and contains no reduced vertices.

We continue this reduction till bad 2-cell is gone.
2nd. We will reduce bad 1-cells $D$ recursively as follows:
2nd. We will reduce bad 1-cells $D$ recursively as follows:

We choose a next 1-cell which is bad in $G|D$ and contains no reduced vertices.
2nd. We will reduce **bad 1-cells** $D$ recursively as follows:

We choose a next 1-cell which is **bad in $G|D$** and contains no reduced vertices.

We continue this reduction till **bad 1-cell is gone**.
3rd. We will reduce **bad 5-cycles** $D$ recursively as follows:

![Diagram](https://via.placeholder.com/150)

We choose a next 5-cycle which is **bad** in $G_{|D}$ and contains no reduced vertices. We continue this reduction till the **bad** 5-cycle is gone.
3rd. We will reduce bad 5-cycles $D$ recursively as follows:

We choose a next 5-cycle which is bad in $G \mid D$ and contains no reduced vertices.
3rd. We will reduce bad 5-cycles $D$ recursively as follows:

We choose a next 5-cycle which is bad in $G|D$ and contains no reduced vertices.

We continue this reduction till bad 5-cycle is gone.
Finally we will obtain a 3-edge-connected cubic graph $G'$ in which every bad cell contains a reduced vertex.
Finally we will obtain a 3-edge-connected cubic graph $G'$ in which every bad cell contains a reduced vertex.

Actually we have to choose 5-cycles which will be reduced much more carefully.
Finally we will obtain a **3-edge-connected cubic graph** $G'$ in which every bad cell contains a reduced vertex.

Actually we have to choose **5-cycles** which will be reduced much more carefully.

By the theorem of Kaiser and Skrekovski, we can obtain a 2-factor $F'$ such that
Finally we will obtain a 3-edge-connected cubic graph $G'$ in which every bad cell contains a reduced vertex.

Actually we have to choose 5-cycles which will be reduced much more carefully.

By the theorem of Kaiser and Skrekovski, we can obtain a 2-factor $F'$ such that

- $F'$ intersects all 3-cut and 4-cut in $G'$. 
Finally we will obtain a 3-edge-connected cubic graph $G'$ in which every bad cell contains a reduced vertex.

Actually we have to choose 5-cycles which will be reduced much more carefully.

By the theorem of Kaiser and Skrekovski, we can obtain a 2-factor $F'$ such that

- $F'$ intersects all 3-cut and 4-cut in $G'$.
- if $F'$ contains a 5-cycle $C$, then $C$ contains a reduced vertex.
From the 2-factor $F'$ of $G'$, we construct a desired even subgraph $F$ of $G$. 
From the 2-factor $F'$ of $G'$, we construct a desired even subgraph $F$ of $G$.

For 5-cycles, e.g.,
From the 2-factor $F'$ of $G'$, we construct a desired even subgraph $F$ of $G$.

For 5-cycles, e.g.,
For 1-cells, e.g.,

(a)  
(b)
For 1-cells, e.g.,

For 2-cells, e.g.,
There is an infinite family of 3-edge-connected cubic graphs in which every dominating even subgraph contains 9-cycles. (Cada, Chiba, Ozeki, Y 2015+)
There is an infinite family of 3-edge-connected cubic graphs in which every dominating even subgraph contains 9-cycles. (Cada, Chiba, Ozeki, Y 2015+)

Problem.
Does a 3-edge-connected cubic graph have a dominating even subgraph in which every cycle contains $\geq 9$ vertices?
There is an infinite family of 3-edge-connected cubic graphs in which every dominating even subgraph contains 9-cycles. (Cada, Chiba, Ozeki, Y 2015+)

**Problem.**
Does a 3-edge-connected cubic graph have a dominating even subgraph in which every cycle contains ≥ 8 vertices?

**Conjecture.**
Any 3-edge-connected graph has a dominating even subgraph in which every component contains ≥ 6 vertices.
Thank you for your attention.