Singular edges and hamiltonicity in claw-free graphs with locally disconnected vertices

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Pilsen, March 29- April 3. 2015

Outline

1. Introduction

2. Further extension of the result above

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A cycle is called a Hamiltonian cycle of a graph G if it has all vertex of G.

Definition 1.2 • A Hamiltonian graph

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A graph is called claw-free if it has no induced subgraph isomorphic to $K_{1,3}$.

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Instance: Graph G.

Question: Does G have a hamiltonian cycle?

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HAMILTON is NP-complete; even for the class of claw-free graphs.

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Definition 1.4 • Locally connected

A vertex v of a graph G is *locally connected* if $G[N_G(v)]$ is connected; otherwise, it is *locally disconnected*.

Definition 1.5 • N_2-locally connected

A vertex v of a graph G is *N_2-locally connected* if the subgraph of G induced by the edge $\{e = xy \in E(G) : v \notin \{x, y\}$ and $\{x, y\} \cap N(v) \neq \emptyset\}$ is connected.

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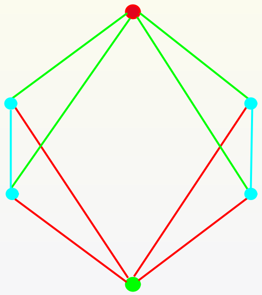
Every locally connected graph is N_2-locally connected, but the converse is not true.



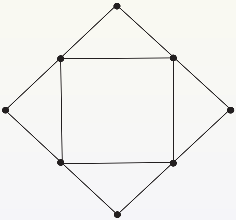
The related theorems

Theorem 1.1 (Oberly and Sumner, 1979)

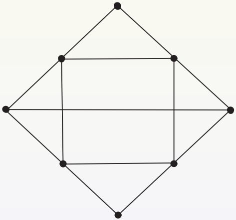
Every connected locally connected claw-free graph on at least three vertices is Hamiltonian.

Theorem 1.2 (Ryjáček, 1990)

Let G be a connected, N_2 -locally connected claw-free graph without vertices of degree 1, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 such that every vertex x of degree 4 in H is locally disconnected in G. Then G is Hamiltonian.



(a) G_1



(b) G_2:

Theorem 1.3 (Bielak, 2000)

Let G be a connected, N_2 -locally connected claw-free graph without vertices of degree 1, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 such that every vertex of degree 3 or 4 in H is locally disconnected in G. Then G is Hamiltonian.

Theorem 1.3 means that the conclusion of Theorem 1.2 is also true for the degree set $\{3, 4\}$ of G_1, G_2.

Note

The later result implies the forward one:

Question:

- Is there a more extending result???
- If yes, how to do it???

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Singular edge

We need the following definitions.

Definition 1.6 • Singular edge

An edge e of G is *singular* if it does not lie on any triangle of G; otherwise, it is *non-singular*.

Definition: $\mathcal{P}(C, k, l)$

$\mathcal{P}(C, k, l)$: A locally disconnected vertex lies on an induced cycle C of length at least 4 that has at most k non-singular edges and has at least $k - l$ locally connected vertices.

Theorem 1.4 (R. Tian and X., 2012^+)

Let G be a connected claw-free graph satisfying the following two conditions:

(i) Every locally disconnected vertex of degree at least 3 lies on an induced cycle of length at least 4 with at most s non-singular edges and with at least $s-3$ locally connected vertices**(**$i.e., \mathcal{P}(C, s, 3)$ **holds)**;

(ii) Every locally disconnected vertex of degree 2 lies on an induced cycle C of length at least 4 with at most s non-singular edges and with at least $s-2$ locally connected vertices**(**$i.e., \mathcal{P}(C, s, 2)$ **holds)** such that $G[V(C) \cap V_2(G)]$ is a path.

Then G is Hamiltonian.

Theorem 1.4 (R. Tian and X., 2012^+)

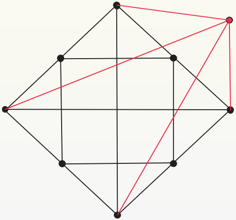
Let G be a connected claw-free graph satisfying the following two conditions:

- Every locally disconnected vertex of degree at least $d \in \{2, 3\}$ satisfies $\mathcal{P}(C, s, d)$;
- If a locally disconnected vertex of degree two satisfies $\mathcal{P}(C, s, 2)$, then $G[V(C) \cap V_2(G)]$ is a path.

Then G is Hamiltonian.

Proposition 1.5 (R. Tian and X., 2012^+)

If a graph G satisfies the condition of Theorem 1.3 and G is not isomorphic to G_3, then G satisfies condition of Theorem 1.4.



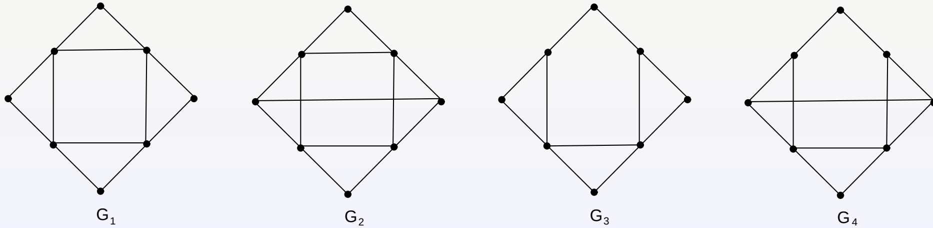
G_3:

Definition 1.5 • N^2-locally connected

A vertex v of a graph G is *N^2-locally connected* if the subgraph of G induced by the vertices set $\{v \in V(G) : 1 \leq d_G(v, x) \leq 2\}$ is connected.

Corollary (Li, 2000)

Every connected N^2-locally connected claw-free graph G with $\delta(G) \geq 2$ that has no induced subgraph isomorphic to G_1, G_2, G_3, G_4 is Hamiltonian.



Remark

The above results show that our result extends both of Bielak's results and Li's results, therefore also extends the results of Ryjáček, and, Oberly and Sumner.

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What happen if we improve the connectivity of considered graphs?

Theorem 1.6 (R. Tian, X. and Z. Niu, 2012^+)

Let G be a 3-connected claw-free graph. If every locally disconnected vertex lies on some induced cycle C of length at least 4 with at most 4 non-singular edges **(**$i.e., \mathcal{P}(s,4)$ **holds))**, then G is Hamiltonian.

The above result extends the following result, since every locally disconnected, N_2-locally connected vertex of a graph G lies on some induced cycle of length 4.

Theorem 1.7 (Lai, Shao and Zhan, 2005;Conjectured by Ryjáček, 1990)

Every 3-connected N_2-locally connected claw-free graph is Hamiltonian.

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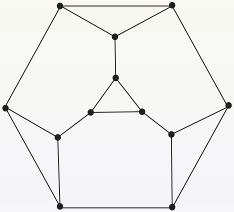
An example

- The graph G in Figure is a connected claw-free graph such that every locally disconnected vertex of G lies on an induced cycle of length 5 with 3 non-singular edges;
- but it is not N_2-locally connected.



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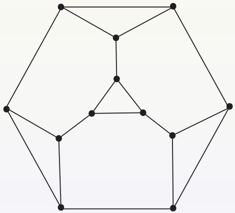

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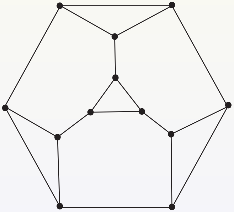

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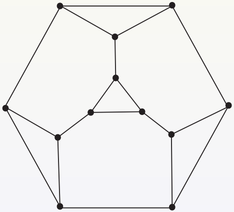

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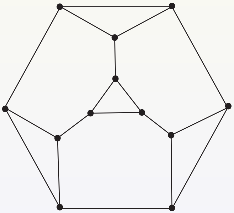

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2 Further extension of the result above

The Matthews and Sumner Conjecture

Conjecture 2.1 (Matthews and Sumner, 1984)

Every 4-connected claw-free graph is Hamiltonian.

Remark

Up to now, we do not know whether there is a weaker condition of our type for those 4-connected graphs.

Conjecture 2.1'

Every 4-connected claw-free graph with the property $\mathcal{P}(4, s, s)$ is Hamiltonian.

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Theorem(Z. Ryjáček, P. Vrána and X., working 2014^+)

Let G be a connected claw-free graph satisfying the following conditions:

(I) every locally disconnected nonsingular vertex of degree 4 is on an induced cycle of length at least 4 with at most 4 nonsingular edges;

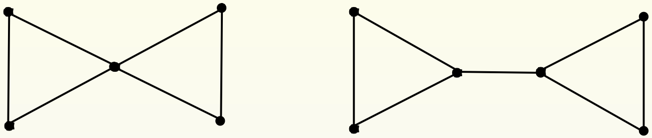
(II) every locally disconnected vertex that is not nonsingular of degree 4 is on an induced cycle of length at least 4 with at most 3 nonsingular edges;

(III) every singular vertex of degree 2 is on an induced cycle C of length at least 4 with at most 2 nonsingular edges such that $G[V(C) \cap V_2(G)]$ is a path or a cycle.

Then either G is hamiltonian or G is the line graph of the graph obtained from $K_{2,3}$ by attaching a pendant edge to every its vertex of degree two.

Corollary

Every connected claw-free graph of $\delta(G) \geq 3$ satisfying Conditions (I) and (II) of the theorem above is hamiltonian.

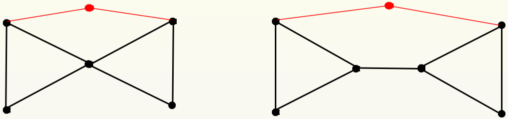


$\Gamma_0 =$ Hourglass, Γ_1,:

$\Gamma_0 = Hourglass$, a graph consisting of two triangles sharing exactly one common vertex.

Γ_1: a graph consisting of two triangles by adding an edge between two vertices of the two triangles.

A graph is said to have *the* Γ_i k*-Property* if for every induced Hourglass Γ_i, there exists a path $P(\Gamma_i)$ joining two vertices of Γ_i whose distance is $i+2$ in Γ_i such that $P(\Gamma_i)$ has no inner vertex of Γ_i and it has at most k nonsingular edges.



Γ_i 2-Property:

Corollary

Let G be a connected claw-free graph with (II) and (III) of the theorem above such that G has the Hourglass 2-property. Then G is hamiltonian or the line graph of $K'_{2,3}$ depicted in the theorem above.

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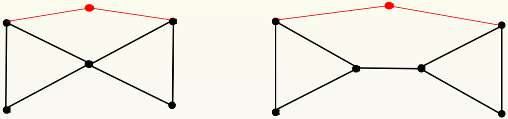

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Corollary

Every 3-edge-connected claw-free graph with Γ_i $(i+2)$-Properties for $i \in \{0,1\}$ is hamiltonian.

Proof idea: Ryjáček's closure + Lai's Theorem + Harary and Nash-Williams's Theorem.

Theorem(Z. Ryjáček, 1997)

If G is a claw-free graph, then there is a closed claw-free graph $cl(G)$ such that

- there is a triangle-free graph H such that $cl(G) = L(H)$;
- G is Hamiltonian if and only if $cl(G)$ is Hamiltonian.

Theorem(H.-J. Lai, 1991)

Let G be a 2-connected graph with $\delta(G) \geq 3$. If every edge of G lies on a cycle of length at most 4, then G has a spanning Eulerian subgraph.

Theorem (Harary and Nash-Williams, 1965)

Let G be a graph with at least 3 edges. Then the line graph $L(G)$ is Hamiltonian if and only if G has a dominating Eulerian subgraph.

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