HAMILTONIAN CYCLES IN SPANNING SUBGRAPHS OF LINE GRAPHS

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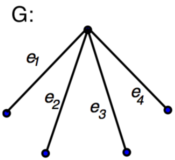
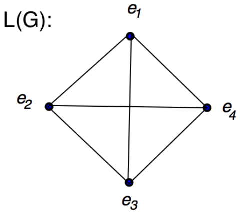
Plzeň, April 2nd, 2015

Line graph:





THEOREM (HARARY AND NASH-WILLIAMS, 1965)

Let G be a graph not a star. Then $L(G)$ is Hamiltonian if and only if G has a dominating closed trail.

Question: If G has a dominating closed trail, can we remove some edges in $L(G)$ s.t. the resulting graph is Hamiltonian?

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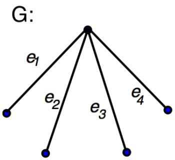

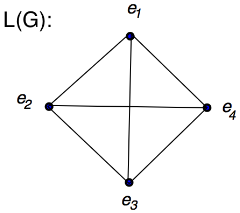

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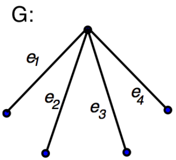

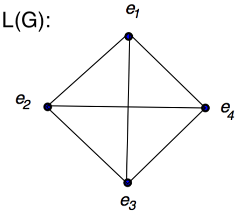

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Theorem (Ryjáček, 1997)

Let G be a claw-free graph. Then

1. *the closure $cl(G)$ is well-defined.*
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DEFINITION OF $SL(G)$

- it's a spanning subgraph of $L(G)$,
- every vertex $e = uv$ is adjacent to at least $min\{d_G(u) - 1, \lceil \frac{3}{4} d_G(u) + \frac{1}{2} \rceil\}$ vertices of $E_G(u)$ and to at least $min\{d_G(v) - 1, \lceil \frac{3}{4} d_G(v) + \frac{1}{2} \rceil\}$ vertices of $E_G(v)$.

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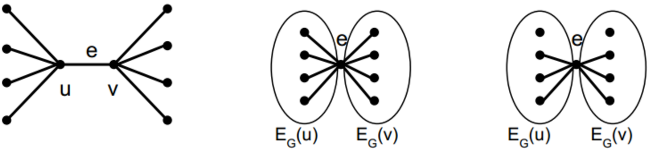

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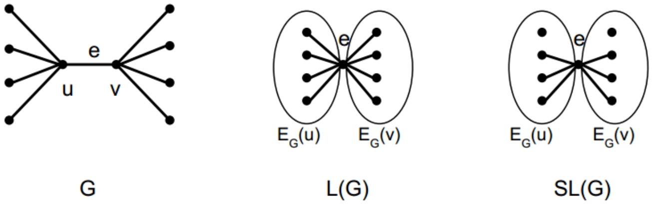

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Theorem (Li, Bai, Yang and He)

If $L(G)$ is Hamiltonian, then every $SL(G) \in \mathcal{SL}(G)$ is also Hamiltonian.

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We need some more notation,

- **Fake edge**: an edge in $L(G)$, not in $SL(G)$.
- **Stable vertex**: Suppose that H is a Hamiltonian cycle of $L(G)$, if e^-, e *and* e^+ are all in the same set $E_G(v)$(v is any vertex of G), e is called a stable vertex of H.

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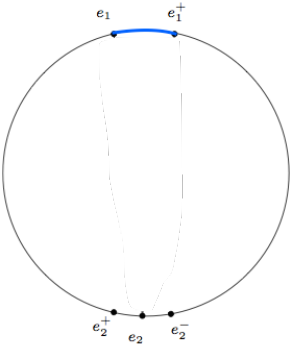

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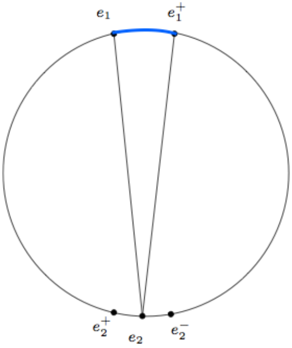

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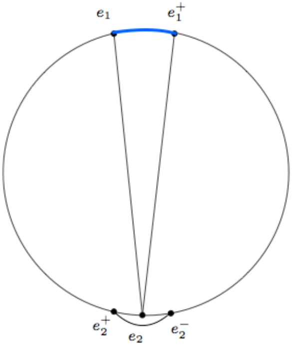

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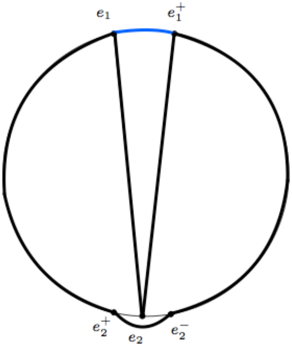

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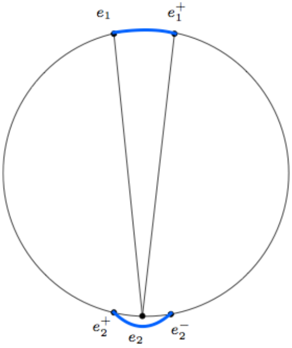

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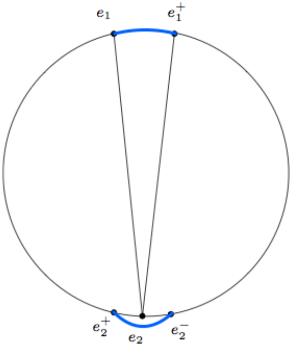

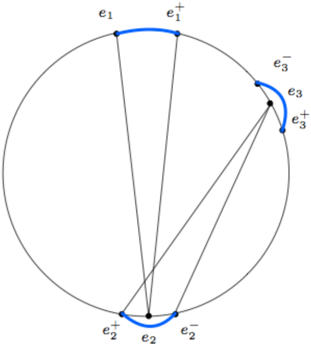

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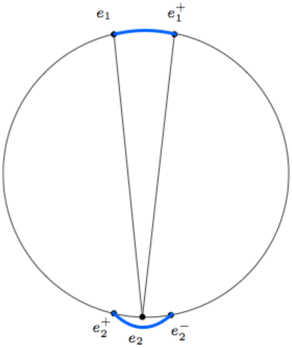
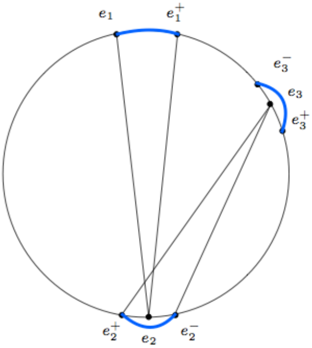




- e_i: **stable vertex.**
- Since the graph is finite, this progress will stop after some steps.

Sketch of the proof

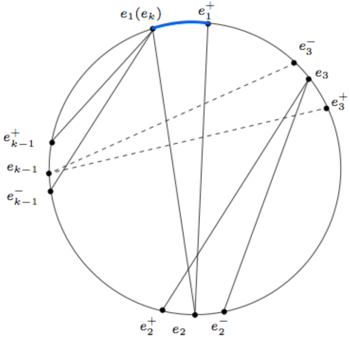
Key lemma

Let $e_1 e_1^+$ be a fake edge in H, e_1 and e_1^+ belong to $E_G(u_1)$ ($u_1 \in V(G)$), there exists a vertex $e_2 \in E_G(u_1)$ such that

1. $e_1 e_2$ and $e_1^+ e_2$ are edges in $SL(G)$,
2. e_2, e_2^-, e_2^+ all belong to $E_G(u_2)$ for a $u_2 \in V(G)$ with $u_2 \neq u_1$,
3. $e_2^- e_2^+$ is a fake edge in $L(G)$ and $e_2^- e_2, e_2 e_2^+$ are non-fake edges in $L(G)$.

Sketch of the proof

- **Case 1.** $e_k = e_1$.



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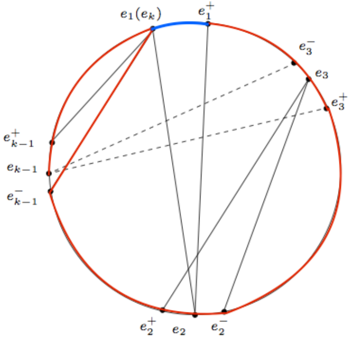

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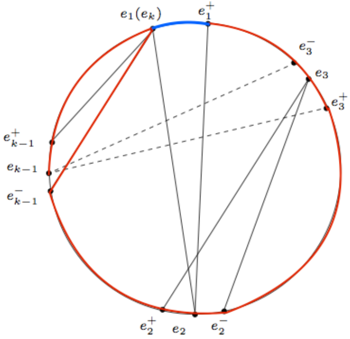

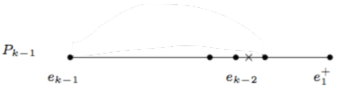

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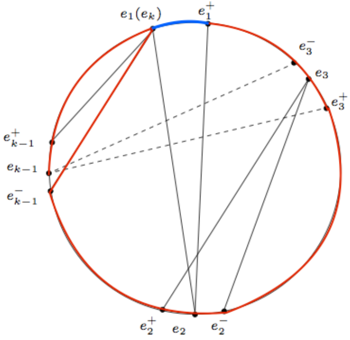

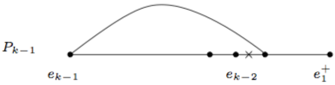

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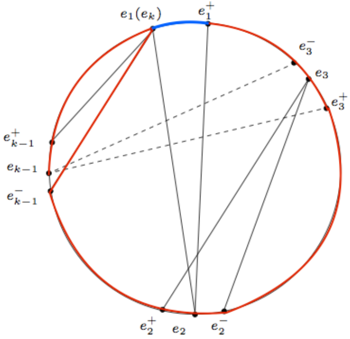

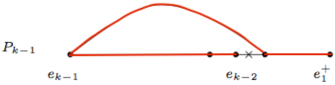

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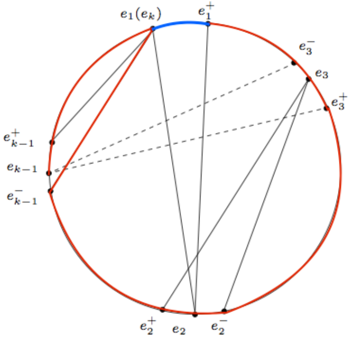

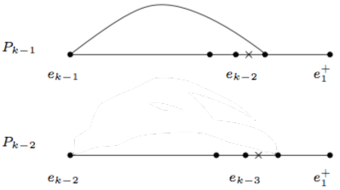

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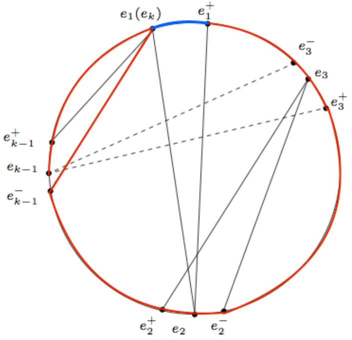

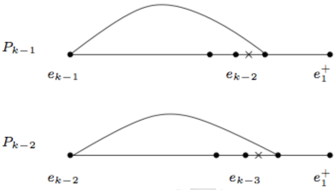

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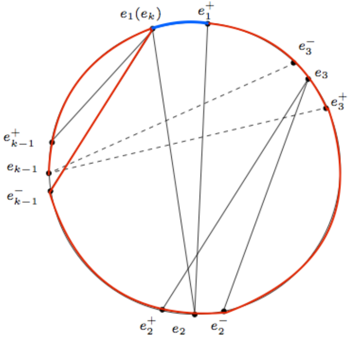

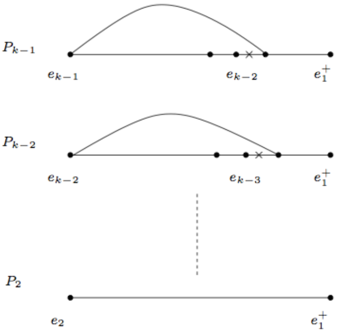

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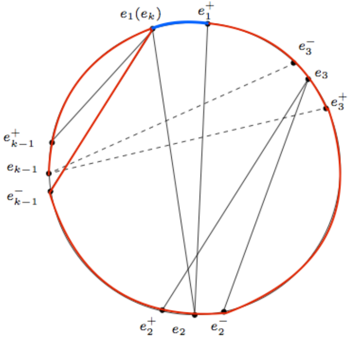

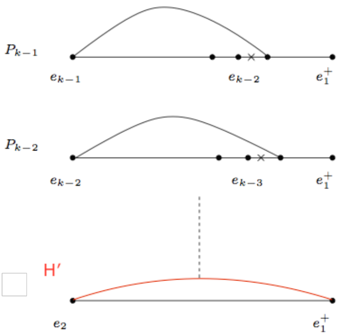

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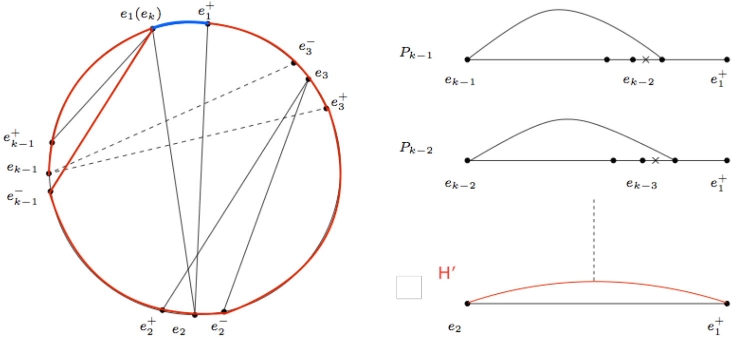

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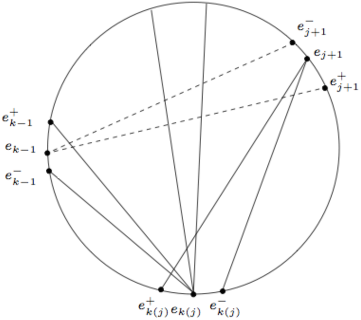

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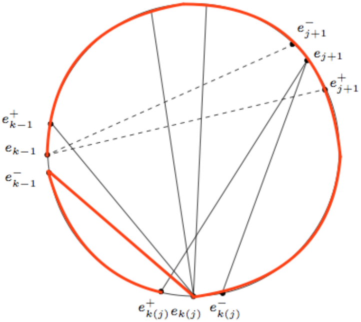

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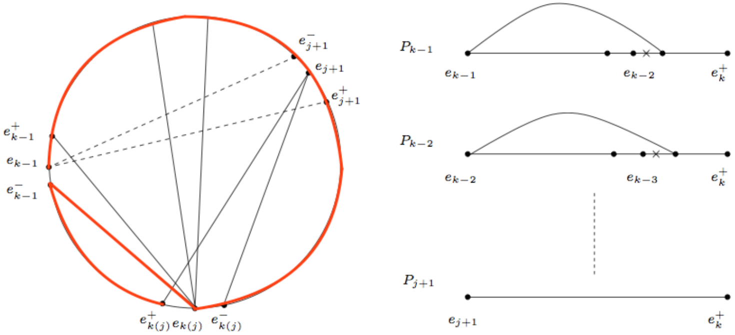

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- Construct a new Hamiltonian cycle in $L(G)$,

$$H^{'} = P_{j+1} + e_k^{+} e_{j+1}.$$

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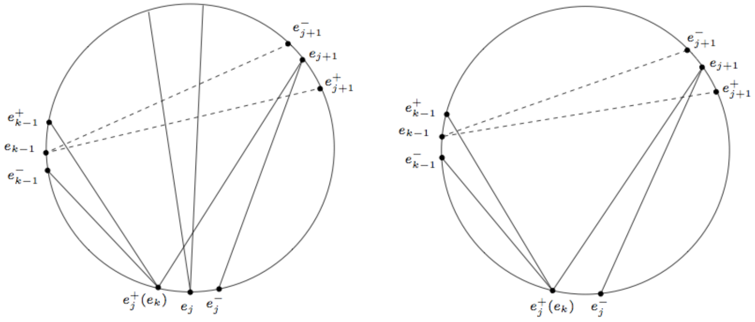

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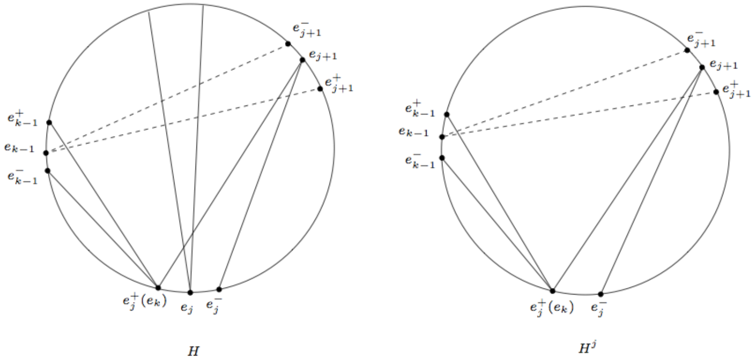

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