# The Paulus-Rozenfeld-Thompson strongly regular graph on 26 vertices: animated logo of WL 2018

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- This graph $T$ was discovered a few times:
  - by A. J. L. Paulus in [14];
  - by M. Z. Rozenfeld in [15];
  - by D. M. Thompson in [16].
- The graph is listed in [17] as graph #8 in the list of all SRGs with the parameters $(v, k, \lambda, \mu) = (26, 10, 3, 4)$.
- The graph $T$ appears also in [6]. As J. J. Seidel is writing in his review MR0505849 of [6], significant references to [1] and [17], regarding graph $T$ and SRGs related to it, were missing in [6].
- For the modern reader, probably the most convenient reference would be to the article Paulus graphs on the home page [4] of Andries Brouwer. There one can find 15 SRGs with the parameters $(25, 12, 5, 6)$ and 10 SRGs with the parameters $(26, 10, 3, 4)$ are considered, including also their adjacency lists. $T$ has name P26.10.
- The graph $T$ has the highest symmetry among all 10 SRGs on 26 vertices in the family of Paulus graphs. The group $G = Aut(T)$ of order 120 is isomorphic to $A_5 \times \mathbb{Z}_2$ and has two orbits of length 20 and 6 on the vertex set $V = V(T)$.

- Thus we get an automorphic equitable partition $\Pi$ of $V(T)$, see [8], [13], [12]. Its collapsed matrix is equal to $\begin{pmatrix} 7 & 3 \\ 10 & 0 \end{pmatrix}$. Here $\Pi = \{C_1, C_2\}$, $|C_1| = 20$, $|C_2| = 6$, the induced graph $\tilde{T}_1 = (C_1, E_{11})$ in $C_1$ is regular of valency 7, while the induced graph $\tilde{T}_2$ on $C_2$ is empty, that is a coclique of size 6.

- Our description of $T$ is given in the terms of the Schurian coherent configuration (CC) (the term goes back to D. Higman [9], see [5]). We consider intransitive action $(A_5, C_1 \cup C_2)$ of the alternating group $A_5$ on the orbits of length 20 and 6 and construct, using computer package COCO [7], the CC $W(A_5, V)$. This CC has rank 14 and consists of two fibers (orbits of $A_5$) and 14 2-orbits of $A_5$ (in the same sense of H. Wielandt [18]). The edge set $E(T)$ is a union of suitable 2-orbits; the idea of description goes back to D. Thompson, though we are using reasonably unified modern language of CCs.

- An essential feature of the graph $T$, in comparison with 9 other SRGs on 26 vertices, is that the used coclique $C_2$ is coherent. This means that fiber, consisting of $C_2$, is not destroyed by the Weisfeiler-Leman stabilization of $T$ together with partition $\Pi$.

- We need to describe the edge set $E(T)$ as union $E_{11} \cup E_{12} \cup E_{22}$. The set $E_{22}$ of the edges in the coclique, induced by $C_2$, is empty. Let us start from $E_{11}$. For this purpose we will investigate an auxiliary structure of dodecahedron, platonic solid $D$ and distance transitive graph (DTG), see [2], [3].

- First, we describe permutation group $(A_5, V) = (A_5, C_1 \cup C_2)$.
  - Clearly, up to similarity, $A_5$ has one transitive action of degree 6 and one of degree 20.
  - We start from the natural action $(A_5, [0, 4])$ of degree 5. Then the induced action of $A_5$ on ordered pairs of elements from $[0, 4]$ has degree 20.
  - Action of degree 6 coincides with the action of $A_5$ on six pentagons as in Figure 1 below.
  - The 14 2-orbits of $(A_5, V(T))$ are described with the aid of representatives:

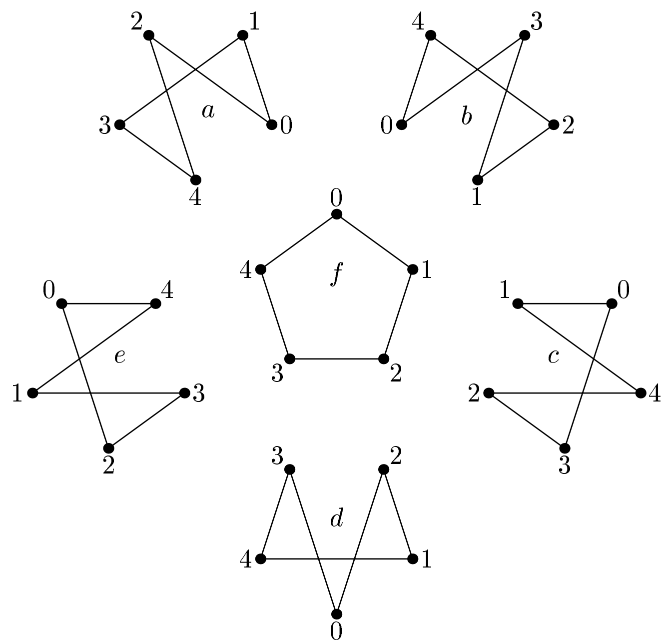
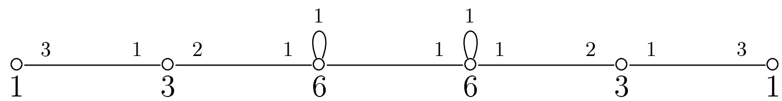



Figure 1: Six pentagons in $\Phi_{5,5}$.

$R_0 = ((0,1),(0,1))^{A_5}$,
$R_1 = ((0,1),(1,2))^{A_5}$,
$R_2 = ((0,1),(2,3))^{A_5}$,
$R_3 = ((0,1),(2,4))^{A_5}$,
$R_4 = ((0,1),(4,0))^{A_5}$,
$R_5 = ((0,1),(0,3))^{A_5}$,
$R_6 = ((0,1),(4,1))^{A_5}$,
$R_7 = ((0,1),(1,0))^{A_5}$,
$R_8 = ((0,1),(0,1,2,3,4))^{A_5}$,
$R_9 = ((0,1),(0,3,1,2,4))^{A_5}$,
$R_{10} = ((0,1,2,3,4),(0,1))^{A_5}$,
$R_{11} = ((0,1,2,3,4),(1,3))^{A_5}$,
$R_{12} = ((0,1,2,3,4),(0,1,2,3,4))^{A_5}$,
$R_{13} = ((0,1,2,3,4),(0,1,3,2,4))^{A_5}$,
Here $R_0$, $R_{12}$ reflexive orbits, $R_4 = R_1^T$, $R_{10} = R_8^T$, $R_{11} = R_9^T$, while other six 2-orbits are symmetric and non-reflexive.

- Clearly, relations from $R_0$ to $R_7$, defined on the set $C_1$, define a homogeneous CC of rank 8. It turns out that the graph $(C_1, R_2)$ is one of the classical cases of DTGs, namely $\Delta$. Its planar diagram is depicted in Figure 2.

- The intersection diagram of $\Delta$ looks as follows:




  Denote by $\Delta_i$, $1 \le i \le 5$, distance $i$ graphs for $\Delta$. They are defined by relations: $R_2$, $R_1 \cup R_4$, $R_5 \cup R_6$, $R_3$, $R_7$, respectively.

- There are three imprimitivity systems for $(A_5, C_1)$. Relations $R_5$ and $R_6$ define graphs $5 \circ K_4$, relation $R_7$ defines antipodal system $10 \circ K_2$. Their combinatorial meaning in terms of natural action $(A_5, [0, 4])$ is evident: pairs share first or second element, or are opposite.

- Note that graph $(C_1, R_3)$ is also dodecahedron, opposite to $\Delta = \Delta_1$.

- Now, we describe the induced subgraph $(C_1, E_{11})$ on 20 vertices: it is defined by the union $R_5 \cup R_6 \cup R_7$ of all imprimitive disconnected basic SRGs of $V(A_5, C_1)$. Its diagram symbolically appears in Figure 3.

  We have $5 \times 5$ grid with removed vertices on the main diagonal. Thick horizontal and vertical lines correspond to 10 cliques of size 4. In addition, each vertex is joined with its mate via reflection with respect to the (empty) main diagonal. Thus, finally, we get regular graph on 20 vertices of valency 7.

- The remaining 60 edges from $E_{12} (= E_{21})$ form bipartite graph with two parts of size 20 and 6 and valencies 3 and 10 respectively. This graph is defined by union $R_8 \cup R_{10}$. The combinatorial meaning, in terms of $(A_5, [0, 4])$ is pretty clear: a pentagon $X$ in $C_2$ is joined by edge with pair $(a, b)$ if and only if $\{a, b\}$ is one of the edges of $X$.

- Thus, in principle, graph $T$ is fully described. It remains, however to provide a few nice diagrams of $T$.

- Recall that $\Delta$ is the skeleton of a dodecahedron $D$. The group of space rotations of orientable map, corresponding to $D$, is isomorphic to $A_5$. The dual map see, e.g. [10] to $D$ is the icosahedron $I$. Twelve vertices

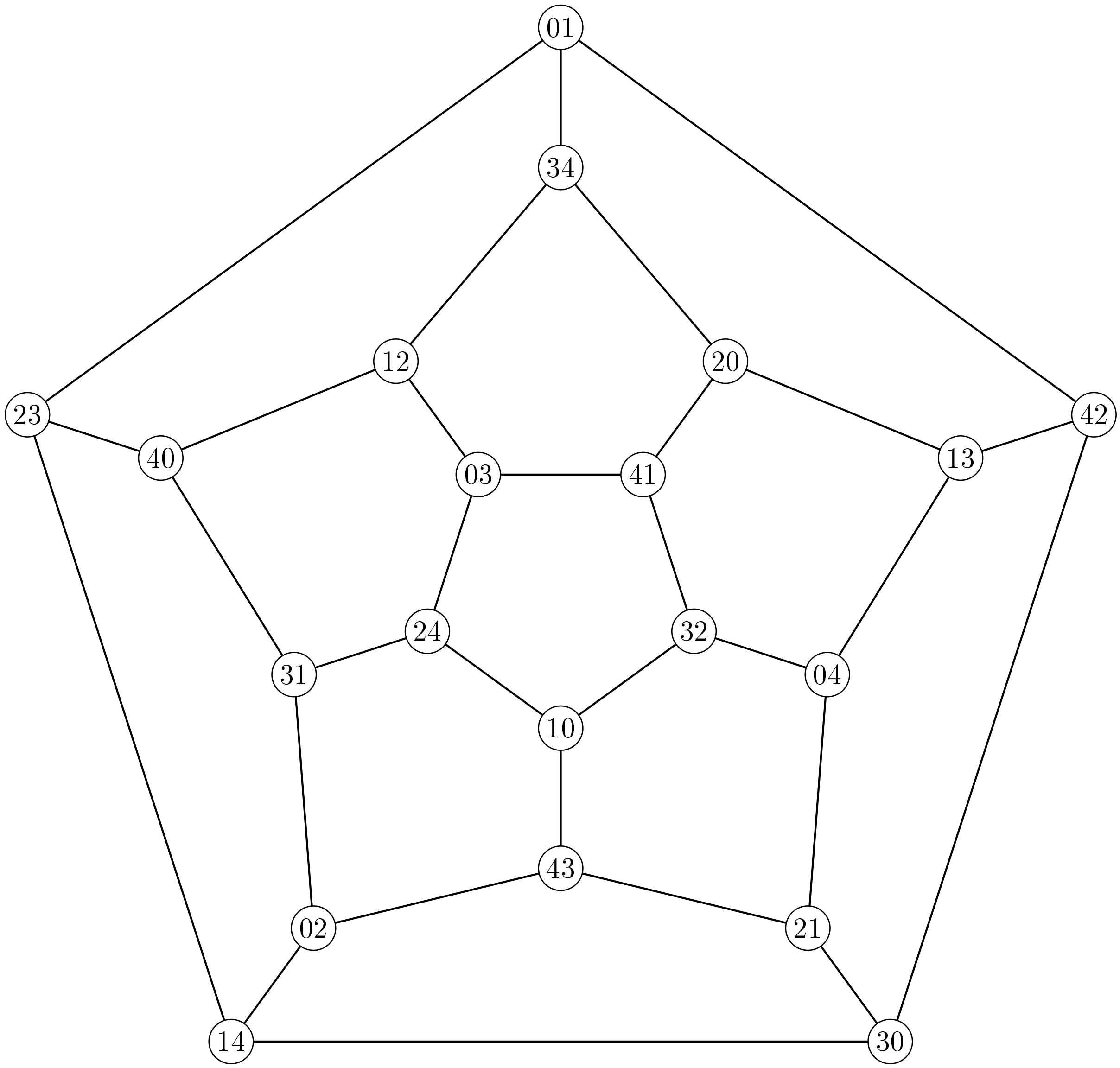


Figure 2: Dodecahedron.

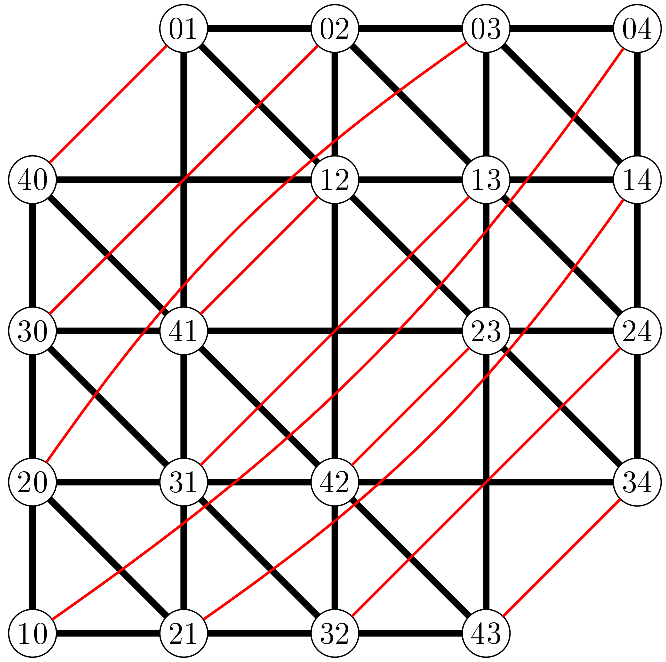



Figure 3: Induced subgraph on $C_1$.

of $I$ correspond to the faces of $D$. The skeleton of $I$ is also DTG, it is antipodal graph of diameter 3. The six space diagonals of $I$, regarded as opposite pairs of vertices, form imprimitivity system $6 \circ K_2$ of the action of $A_5$ on the vertex set of $I$. These six diagonals bijectively correspond to the set $C_2 = \{a, b, c, d, e, f\}$.

- Now we can provide to the reader an an explanation cf. [11] of our animated description of $T$.
  - start from the canvas of planar diagram of $\Delta$;
  - find six representatives of antipodal faces of $D$;
  - denote them by the elements of $C_2$, see Figure 4.
  - put six extra vertices.
  - depict edges from $\Delta_3$, where $\Delta_3$ is the distance-3 graph of $\Delta$.
  - add edges from $\Delta_5$, where $\Delta_5$ is the distance-5 graph of $\Delta$.
  - join each element $X$ of $C_2$ with the vertices of the pentagon surrounding $X$, as well as with the opposite pentagon of $\Delta$.
  - Different colors mean different types of edges.
- According to construction, $Aut(T) \geq A_5 \times \mathbb{Z}_2$, the group, isomorphic to $Aut(\Delta)$.
- It is an easy exercise to show that $Aut(T) = Aut(\Delta) \cong A_5 \times \mathbb{Z}_2$.

An animated version is here.

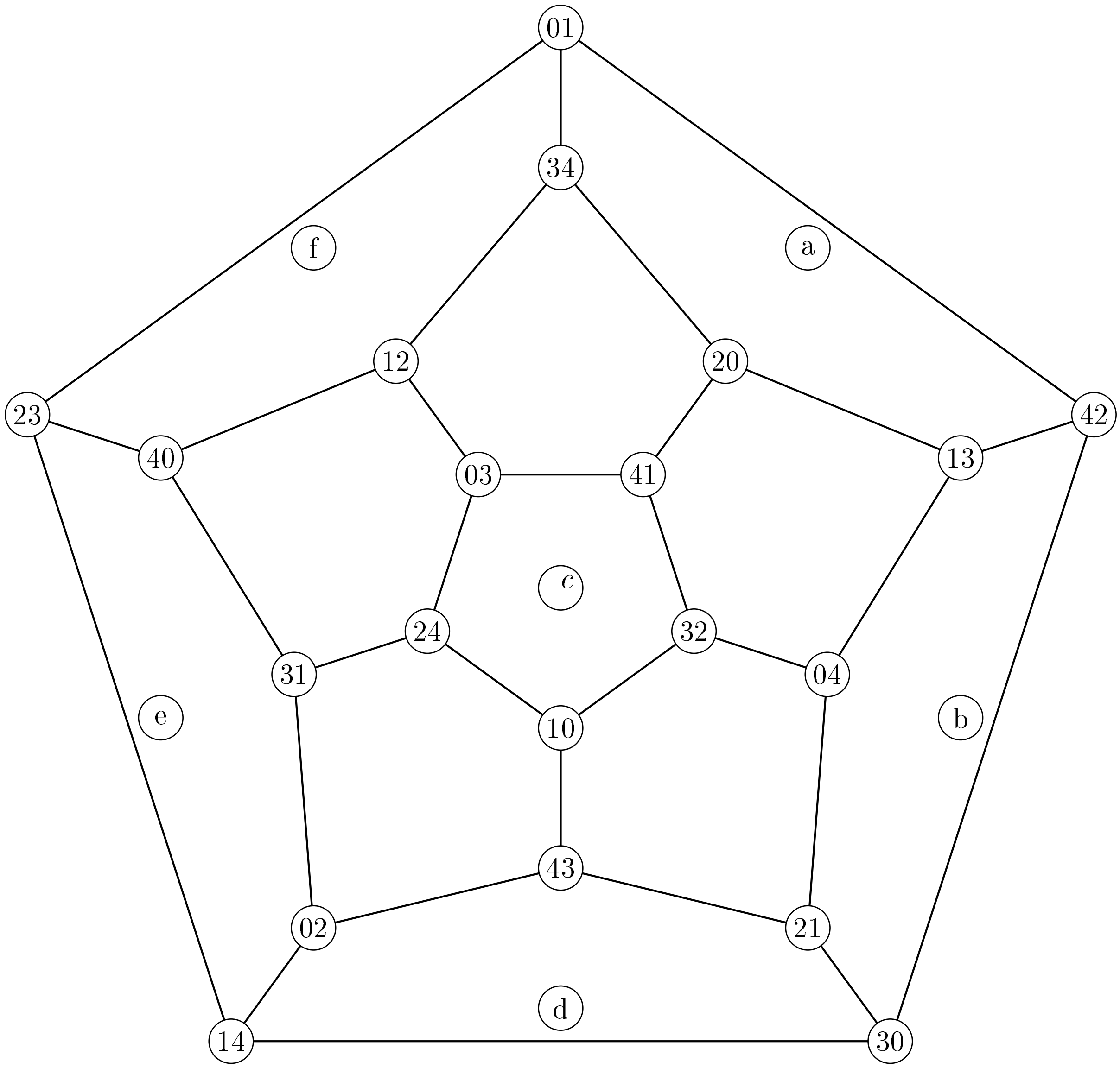



Figure 4: Dodecahedron with six pentagons.

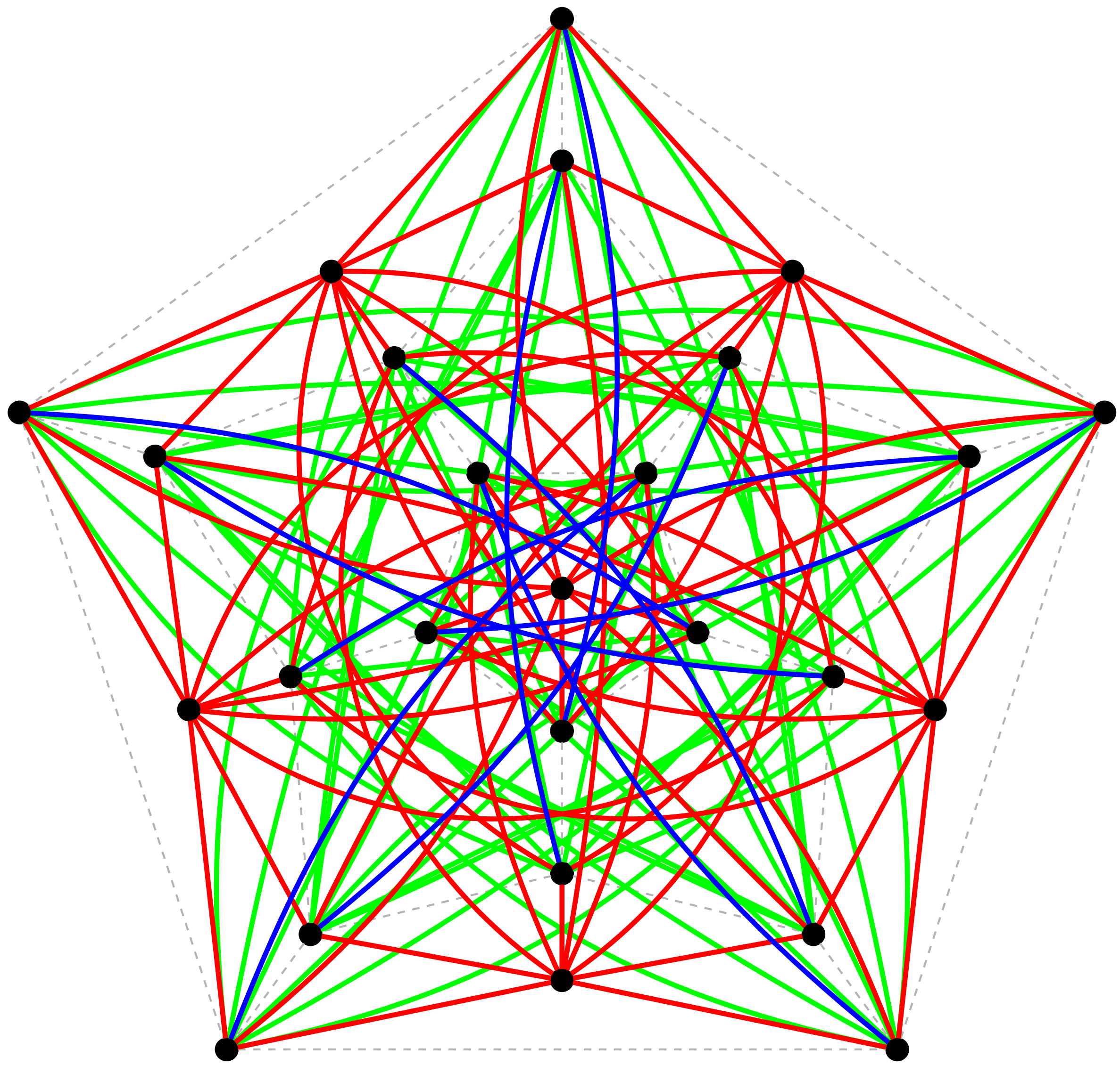


Figure 5: The graph $T$.

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