Symmetry vs. Regularity

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regularity — local, easy to verify
symmetry — global, hard to verify

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regularity: **combinatorial relaxation** of symmetry

$$\text{symmetry} \implies \text{regularity}$$

$$\text{regularity} \overset{?}{\implies} \text{symmetry}$$

Symmetry vs. Regularity

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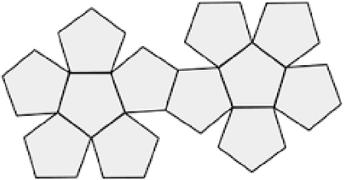
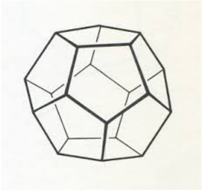
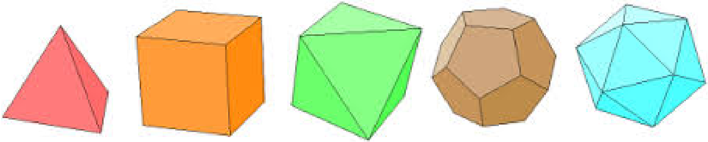




Symmetry vs. Regularity

symmetry $\Longrightarrow$ regularity

regularity $\Longrightarrow$ classification $\Longrightarrow$ symmetry



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  regularity $\stackrel{?}{\Longrightarrow}$ similar structural consequences

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  high degree of symmetry $\Longrightarrow$
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  high regularity $\stackrel{?}{\Longrightarrow}$ few symmetries
- **regularity $\Longrightarrow$ symmetry**
  — **without classification ?**

The challenge

- Consequences of **symmetry** $\Leftarrow$ **group theory**
  often via
  Classification of Finite Simple Groups (CFSG)

- Consequences of **regularity** $\Leftarrow$ what techniques?
  combinatorics, linear algebra, ??
  doing group theory without the groups

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CFSG as magic box

this is not how we
used to do math

Symmetry: quality vs. quantity

Theorem (CFSG + Curtis, Kantor, Seitz (1976))

G doubly trans, $G \neq A_n, S_n \implies |G| \leq n^{1+\log_2 n}$

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Dividend of elementary proof: ideas central to

Theorem (Helfgott–Seress (2013))

$\mathrm{diam}(S_n) < \exp((\log n)^c)$ — *quasipolynomial bound*

$\mathrm{diam}(G)$: max word length under worst generators

More symmetry paradox: graphs with symmetry

X **vertex-transitive** if $\mathrm{Aut}(X)$ transitive on vertices
X **arc-transitive** if $\mathrm{Aut}(X)$ transitive on adjacent pairs

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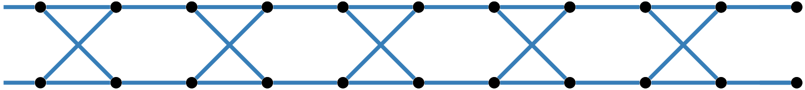
connected 3-regular vertex-transitive graph can have exponentially many automorphisms:
the crossed ladder has $(n/2) \cdot 2^{n/4}$



The crossed ladder (wraparound image)
a 3-regular vertex-transitive graph

Theorem (Tutte (1947))

A connected 3-regular **arc-transitive** *graph has at most* $32n$ *automorphisms.*

incidence geometry: $\mathcal{G} = (P, L, I)$

P – set of "points"

L – set of "lines"

$I \subseteq P \times L$ – incidence relation

Steiner 2-(v,k)-design:

$v = |P|$

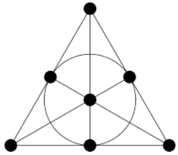
k – "length" of each line (# points incident with line)

Axiom 1: $(\forall x \neq y \in P)(\exists!$ line through $x, y)$

Finite projective plane:

Axiom 2: every pair of lines intersects

$\implies \quad n = k^2 - k + 1$



Symmetry vs. regularity

Theorem (Ostrom–Wagner 1959/65)

If a finite projective plane $\mathcal{G}$ with n points has a doubly transitive automorphism group then it is Desarguesian and therefore $|\operatorname{Aut}(\mathcal{G})| \le O(n^4 \log n)$

Conjecture

If $\mathcal{G}$ is a finite projective plane with n points then

$$|\operatorname{Aut}(\mathcal{G})| < n^C$$

Best known without symmetry:

$$|\operatorname{Aut}(\mathcal{G})| < n^{4+\log_2 \log_2 n}$$

Steiner 2-designs

Steiner 2-(v,k)-design:

$v = |P|$

k – “length” of each line (# points incident with line)

Axiom 1: $(\forall x \neq y \in P)(\exists!$ line through $x, y)$

Examples:

points and lines of d-dim affine and projective geometries over $\mathbb{F}_q$

— these have doubly transitive automorphism groups

$$\Longrightarrow \qquad |\,\mathrm{Aut}\,| < n^{1+\log_2 n}$$

What can be said without the symmetry?

Steiner 2-designs

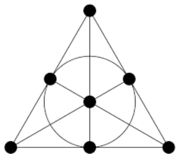
STS: **Steiner triple system:**

Steiner $2-(v,3)$-design: lines have 3 points

Easy to show:

$$|\,\mathrm{Aut}(STS)| < n^{1+\log_2 n}$$

(b/c STS has $\log_2 n$ generators)



Theorem (B-Wilmes, Chen-Sun-Teng (2013))

$\mathcal{G}$ Steiner 2-design with n points $\implies$

$$|\,\mathrm{Aut}(\mathcal{G})| < n^{O(\log n)}$$

Johnson graphs

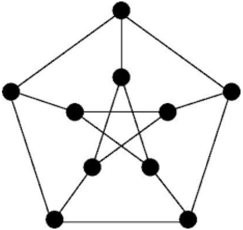
Distance-transitive graphs with many automorphisms

Johnson graph $J(k,t)$: $\quad n = \binom{k}{t} \quad (t < k/2)$

vertices: t-subsets of a k-set

adjacency: intersection = $t - 1$

$\mathrm{Aut}(J(k,t)) = S_k^{(t)}$
induced action of S_k
on t-sets



Petersen = complement of $J(5,2)$

Distance-transitive vs. distance-regular

$$\text{distance-transitive} \implies \text{distance-regular}$$
$$\not\Longleftarrow$$

In fact, $\quad \text{distance-regular} \not\Longrightarrow |\mathrm{Aut}(X)| > 1$

Distance-regular graphs with no automorphisms

Strongly regular graph: distance-regular graph of diameter 2

Theorem (B, Cameron $\sim$ 1980)

$\exists \approx n^{n/2}$ SR graphs with $\leq n$ vertices and no automorphisms.

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"line-graphs" of Steiner triple systems:
vertices: lines, adjacency: intersection

Latin square graphs

Theorem (B, Cameron $\sim$ 1980)

Almost all Latin squares and almost all STSs with $\leq n$ cells/points have no automorphisms.

Distance-regular graphs with no automorphisms

Strongly regular graph: distance-regular graph of diameter 2

Theorem (B, Cameron $\sim$ 1980)

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Open problem: $\exists$ distance-regular graphs
of large diameter
with no automorphisms?
with more than 10 vertex orbits?

If this fails: example of **regularity $\Longrightarrow$ symmetry**

Regularity $\implies$ symmetry?

$X = (V, E)$ graph, $A \subseteq V$
induced subgraph $X(A)$: vertex set A, adjacency: as in X

(symmetry) $X = (V, E)$ **k-homogeneous** if $(\forall A, B \subseteq V)$ if $|A|, |B| \leq k$ and $X(A) \cong X(B)$ then $(\exists \sigma \in \mathrm{Aut}(X))(A^\sigma = B)$

(regularity) $X = (V, E)$ **k-set-regular** if $(\forall A, B \subseteq V)$ if $|A|, |B| \leq k$ and $X(A) \cong X(B)$ then A and B have the same number of common neighbors

$k = 1$: regular graph
$k = 2$: strongly regular

$$k\text{-homogeneous} \implies k\text{-set-regular}$$

Regularity $\implies$ symmetry?

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6-set-regular $\implies$ 6-homogeneous
(in fact k-homogeneous for all k)

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Actual result:

Theorem (Cameron, Klin–Gol'fand (1980))

6-set-regular $\implies$ union of cliques of equal size, or the pentagon, or $L(K_{3,3})$, or their complement

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regularity $\implies$ classification $\implies$ symmetry

Regularity $\Longrightarrow$ symmetry w/o classification?

Perhaps there is such a result.
We shall define certain type of

hidden irregularity

of which the **opposite** is not merely “hidden regularity,” but

hidden **robust symmetry**

Objects with robust (indestructible) symmetry:

Johnson graphs

Primitive groups with a bounded suborbit

Suborbit of $G \le S_n$: orbit of stabilizer
Subdegrees: lengths of suborbits
Sims Conjecture

Theorem (Cameron–Praeger–Saxl–Seitz (CFSG))

Primitive group $G \le S_n$ with subdegree $k \ne 1 \implies |G| \le f(k) \cdot n$.

Corollary: Primitive group, one subdegree $\ne 1$ bounded $\implies$ all subdegrees bounded.

Combinatorial relaxation:

Conjecture (B)

Primitive coherent configuration; degrees of constituents $1 = d_1 \le d_2 \le \cdots \le d_r \implies d_r \le f(d_2)$.

Obvious if r bounded.

Distance-regular vs. distance-transitive graphs

Theorem

Bounded degree $\Longrightarrow$ *bounded size*

1982: Proved for **distance-transitive graphs**
Macpherson–Cameron, based on Sims conj. (CFSG)

Bannai–Ito conjecture: same for **distance-regular** graphs

Distance-regular vs. distance-transitive graphs

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2015: confirmed!
S. Bang–A. Dubickas–J. H. Koolen–V. Moulton
combinatorial lemma by A. A. Ivanov
plus 40 pages of spectral arguments

High regularity, few symmetries

Strongly regular graphs with large automorphism groups:

* trivial (union of cliques, complements) $|\,\mathrm{Aut}\,| > (\sqrt{n})!$
* $H(k,2)$ Hamming graph: $n = k^2$ vertices: $[k] \times [k]$
  adjacent if share a coordinate
  $|\,\mathrm{Aut}(H(k,2))| = 2(k!)^2 \approx n^{\sqrt{n}}$
* $J(k,2)$ Johnson graph: $n = \binom{k}{2}$
  $|\,\mathrm{Aut}(J(k,2))| = k! \approx n^{\sqrt{n/2}}$

"Standard exceptions": trivial SR graphs, Hamming,
Johnson, their complements

High regularity, few symmetries

"Standard exceptions": trivial SR graphs, Hamming, Johnson, their complements

Conjecture (B: quasipolynomial bound)

With these exceptions, SR graphs satisfy
$$|\operatorname{Aut}(X)| \le \exp((\log n)^c)$$

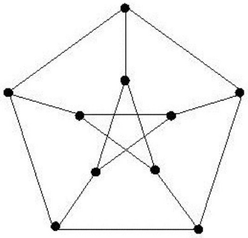
Best known:

Theorem (Chen, Sun, Teng (2013))

With these exceptions, SR graphs satisfy
$$|\operatorname{Aut}(X)| \le \exp(n^{9/37})$$

Individualization/refinement (I/R) heuristics

Irregularity helps refute isomorphism
keep automorphisms down
create/propagate irregularity



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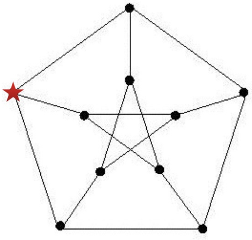

individualize vertex

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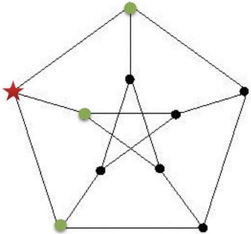

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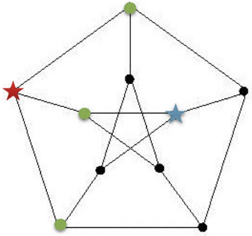

individualize vertex , refine
individualize second vertex

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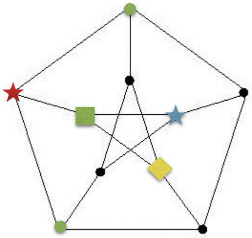

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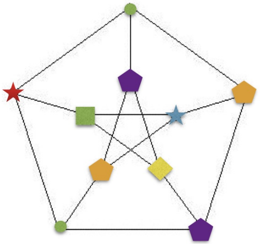

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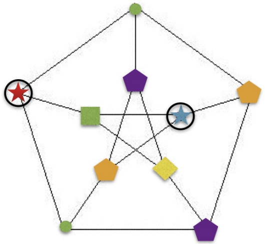

individualize vertex , refine
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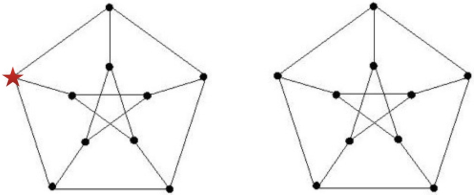
Individualization: multiplicative cost

Individualized $x \in X$: new structure X_x

$$\mathrm{ISO}(X, Y) = \bigcup_{y \in Y} \mathrm{ISO}(X_x, Y_y)$$

multiplicative cost: n



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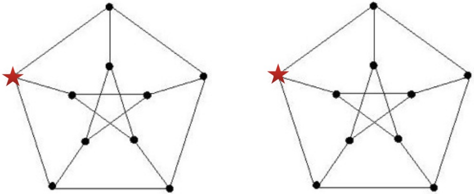

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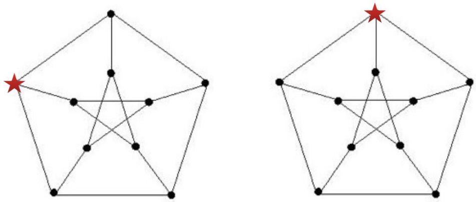

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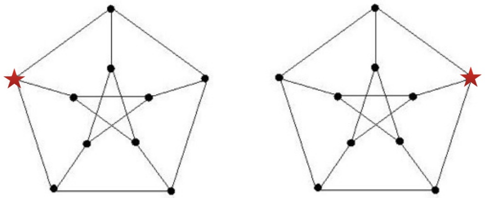

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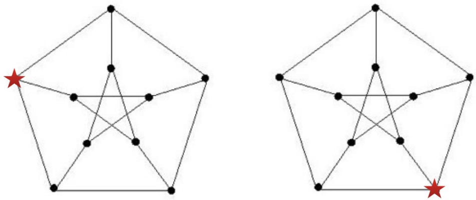

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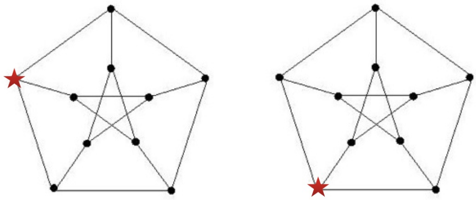

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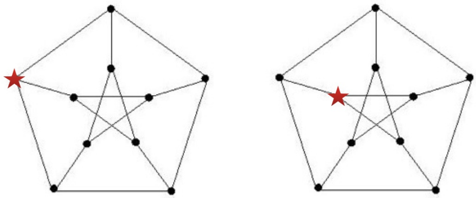

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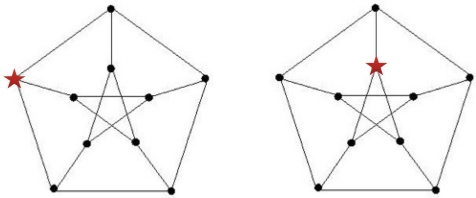

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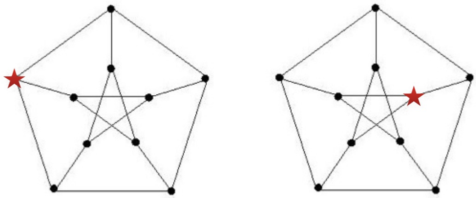

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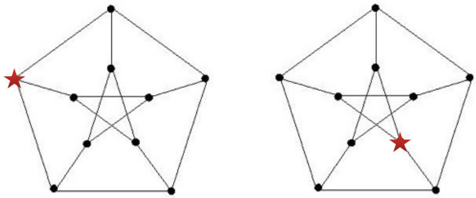

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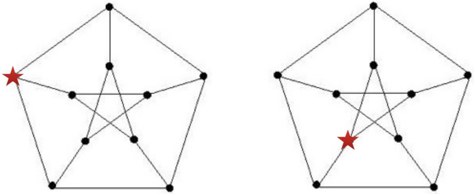


Irregularity helps

Create irregularity: **individualize** t vertices:
give them unique colors

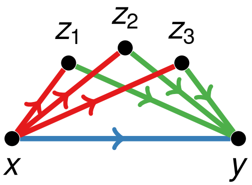
Spread irregularity: **refine coloring**:
count colors of neighbors

multiplicative cost: n^t

branching factor: # instances in canonical set

Advanced refinement: Weisfeiler-Leman 1968

color all ordered pairs, refine by counting triples with shared base and same color composition



coherent configurations: stable under WL

SR graphs: stable for WL (no refinement made)

I/R heuristic

- individualize t points
- canonical refinement

complete split: each point gets unique color

Fact (automorphism bound)

If structure $\mathfrak{X}$ completely splits after t individualizations and canonical refinement them

$$|\operatorname{Aut}(\mathfrak{X})| \leq n^t$$

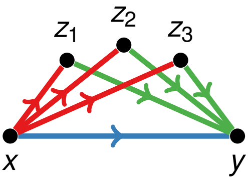
Coherent configurations

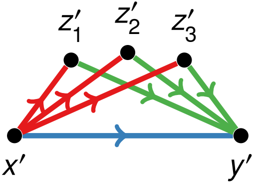
Coherent configuration of rank r: coloring (partition) of
$V \times V = R_0 \dot\cup \ldots \dot\cup R_{r-1}$

diag colors $\neq$ off-diag colors $\mathrm{diag}(V) = \{(x,x) \mid x \in V\}$

color of $x \to y$ determines color of $x \leftarrow y$

for any $(x,y) \in R_k$
p_{ij}^k : $\#\, z$ s.t. $(x,z) \in R_i$ and $(z,y) \in R_j$





Coherent configurations

Primitive coherent configuration:

all vertices (diagonal) same color
every off-diagonal color (strongly) connected

Schurian case: CC primitive $\Leftrightarrow$ group primitive

If rank 3: SR graph or SR tournament (oriented clique)

Theorem (B 1981)

$\mathfrak{X}$ *primitive coherent config of rank* ≥ 3
$\Rightarrow |\mathrm{Aut}(\mathfrak{X})| \leq \exp(\widetilde{O}(\sqrt{n}))$

($\widetilde{O}$ notation hides polylogarithmic factors)

Same paper: SR tournaments: $|\mathrm{Aut}(\mathfrak{X})| \leq n^{O(\log n)}$

Idea of proof

$\mathfrak{X} = (V; R_0, \ldots, R_{r-1})$
Edge colors: $c(x, y) = i$ if $(x, y) \in R_i$

Theorem (B 1981)

$\mathfrak{X}$ primitive coherent config of rank ≥ 3
$\Rightarrow |\mathrm{Aut}(\mathfrak{X})| \leq \exp(\widetilde{O}(\sqrt{n}))$

For $x \neq y \in V$ **distinguishing set**

$$D(x, y) = \{z : c(z, x) \neq c(z, y)\}$$

Fact. If $T \subset V$ intersects each $D(x, y)$ then T **base** of $\mathrm{Aut}(\mathfrak{X})$, i.e., $\mathrm{Aut}(\mathfrak{X})_{(T)} = 1$ (pointwise stabilizer) and so
$$|\mathrm{Aut}(\mathfrak{X})| < n^t \qquad (n = |V|, t = |T|)$$

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$$|\mathrm{Aut}(\mathfrak{X})| < n^t \qquad (n = |V|, t = |T|)$$

So we need $|T| \leq \widetilde{O}(\sqrt{n})$.

Distinguishing number $D = D(\mathfrak{X}) = \min_{x \neq y} |D(x,y)|$

Fact. $T := O((n/D)\log n)$ random points work (whp)

Core technical result:

Theorem

$(\forall x \neq y \in V)(|D(x,y)| \geq \sqrt{n}/2)$

Coherent configurations

Theorem (B 1981)

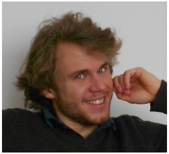
$\mathfrak{X}$ primitive coherent config of rank ≥ 3

$$\Rightarrow |\mathrm{Aut}(\mathfrak{X})| \leq \exp(\widetilde{O}(n^{1/2}))$$

CHALLENGE: reduce bound WNE

(with known exceptions)

Coherent configurations



John Wilmes

Xiaorui Sun

Theorem (Xiaorui Sun - John Wilmes 2015)

$\mathfrak{X}$ *primitive coherent config of rank* ≥ 3

$$\Rightarrow |\mathrm{Aut}(\mathfrak{X})| \leq \exp(\widetilde{O}(n^{1/3})) \text{ \textit{WNE}}$$

- developed structure theory for PCCs
- "clique geometry" separates exceptions

Primitive groups: $|G| \leq n^{O(\log n)}$ WNE
[Cameron 1981, CFSG]
$|G| \leq n^{1+\log_2 n}$ [Maróti 2010, CFSG]
EXCEPTIONS: unique min normal subgroup
(socle) $\mathrm{Soc}(G) = A_m \times \cdots \times A_m$
product of induced A_m actions: $n = \binom{m}{k}^r$
"Cameron groups"
acts on **Cameron schemes:** Johnson/Hamming hybrid

Theorem (Cameron)

Primitive groups: $|G| \leq n^{O(\log n)}$ *WNE*

Ultimate goal:

combinatorial relaxation of this result

Primitive coherent configurations

Conjecture

If $\mathfrak{X}$ primitive coherent configuration not a Cameron scheme, then

(a) $|\mathrm{Aut}(\mathfrak{X})|$ quasipolynomially bounded

(b) polylog individualizations + efficient canonical refinement completely split $\mathfrak{X}$

Primitive coherent configurations

Conjecture

If $\mathfrak{X}$ primitive coherent configuration
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(b) polylog individualizations + efficient canonical refinement completely split $\mathfrak{X}$

Confirmed

- with $n^{1/2}$ indiv [B 1981]
- with $n^{1/3}$ indiv [Sun - Wilmes 2015]

Strongly regular graphs

Conjecture (B: quasipolynomial bound)

With these exceptions, SR graphs satisfy

$$|\operatorname{Aut}(X)| \leq \exp((\log n)^c)$$

Maybe conjecture false.

Conjecture (B: quasipolynomial bound)

With these exceptions, SR graphs satisfy

$$|\mathrm{Aut}(X)| \leq \exp((\log n)^c)$$

Maybe conjecture false.

But if not the log-order, something is *polylogarithmic* about $\mathrm{Aut}(X)$

Definition

Thickness of G: $\theta(G)$: largest t such that A_t is involved in G as quotient of subgroup

$$N \triangleleft H \leq G \qquad H/N \cong A_t$$

Example: X connected graph of degree $\leq d$
$G =$ edge-stabilizer of $\mathrm{Aut}(X)$

$$\Rightarrow \quad \theta(G) \leq d - 1$$

$\rightarrow$ Luks's polynomial-time algorithm
to test **isomorphism of graphs of bounded degree.**

Thickness versus order

Definition

Thickness of G: $\theta(G)$: largest t such that A_t is involved in G as quotient of subgroup

Theorem (B-Cameron-Pálfy 1982 & refinements by Pyber, Liebeck-Shalev)

$G \leq S_n$ *primitive,* $\theta(G) = t \quad \Rightarrow \quad |G| \leq n^{O(t)}$.

Theorem (B 2014)

If X is a non-trivial, non-graphic SRG then

$$\theta(\mathrm{Aut}(X)) = O(\log n)$$

If X graphic (Johnson or Hamming of rank 3) then $\theta(\mathrm{Aut}(X) = \Theta(\sqrt{n})$

Thickness of Aut(SRG)

Theorem (B 2014)

If X is a non-trivial, non-graphic SRG then

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[Pyber 2014] uses Thm for elementary proof of **quasipolynomial** bound on order of **rank-3 groups**

Minimal degree of permutation group

DEF **Minimal degree** of a perm group $G \leq S_n$ is the min # elements moved by any nonidentity element of G.

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Theorem (Bochert 1892)

Min deg of doubly trans group $\neq A_n, S_n$ is $\geq (n-1)/4$.

DEF **Minimal degree** of a perm group $G \leq S_n$ is the min # elements moved by any nonidentity element of G.

Theorem (Bochert 1892)

Min deg of doubly trans group $\neq A_n, S_n$ is $\geq (n-1)/4$.

Theorem (Liebeck 1984 (CFSG))

Min deg of primitive group $\Omega(n/\log n)$ WNE

Min degree vs. thickness

Theorem (Wielandt, 1934)

If min degree of $G \leq S_n$ is $\Omega(n)$ then $\theta(G) = O(\log n)$.

Thickness of Aut(SRG)

Theorem

If X is a non-trivial, non-graphic SRG then

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Theorem

If X is a non-trivial, non-graphic SRG then

$$\theta(\mathrm{Aut}(X)) = O(\log n)$$

Follows by combining Wielandt's bound with

Theorem (B 2014)

If X is a non-trivial, non-graphic SRG then min degree of $\mathrm{Aut}(X)$ *is* $\geq n/8$.

Spectral bound on minimal degree

Theorem (B 2014)

If X is a non-trivial, non-graphic SRG then min degree of $\mathrm{Aut}(X)$ *is* $\geq n/8$.

Conjecture (B)

If $\mathfrak{X}$ is a primitive coherent configuration and not a Cameron scheme then min deg of $\mathrm{Aut}(X)$ is $\geq \Omega(n)$.

Verified for rank 3 above

Verified for rank 4: Bohdan Kivva 2018

NEW!

Spectral bound on minimal degree

X: regular graph of degree k

adjacency matrix $A_X = (a_{ij})_{n\times n}$: $a_{ij} = 1$ if $i \sim j$; o/w $a_{ij} = 0$

adjacency eigenvalues $k = \lambda_1 \geq \cdots \geq \lambda_n \quad \sum \lambda_i = 0$

zero-weight spectral radius: $\xi = \max\{|\lambda_2|, |\lambda_n|\}$

k degree

ξ zero-weight spectral radius

q max number of common neighbors of pairs of vertices

Theorem (B 2014)

Min degree of Aut(X) *is at least*

$$\left(1 - \frac{q+\xi}{k}\right) \cdot n$$

Spectral separation of “bad” graphs from crowd

Union of cliques has $\lambda_n = -1$
Line graphs have $\lambda_n = -2$

Theorem (J. J. Seidel 1968)

If $n \geq 29$ and SRG X has least eigenvalue $\lambda_n \geq -2$ then X is trivial or graphic.

add to this:

Fact: if X SR and $k < (n-1)/2$ then
all eigenvalues are **integers**

$\therefore \quad \lambda_n \leq -3$ in the cases of interest

Spectral separation of “bad” graphs from crowd

ξ: zero-weight spectral radius
q: max $\#$ common neighbors of pair of vertices
k: degree

Theorem (B 2014)

Min degree of Aut(X) *is at least*

$$\left(1 - \frac{q+\xi}{k}\right) \cdot n$$

For this to be useful, we need: $q + \xi \leq 0.99k$

But for SRG: $\lambda_2 |\lambda_n| = k - \mu < k$
so if $\lambda_n \leq -3$ then $\xi = \lambda_2 \leq k/3$.
Bounding q: [B 1980]

For $k = o(n)$ we have $q = o(k)$ [Neumaier 1979, Spielman 1996]

Effect of small zero-weight spectral radius

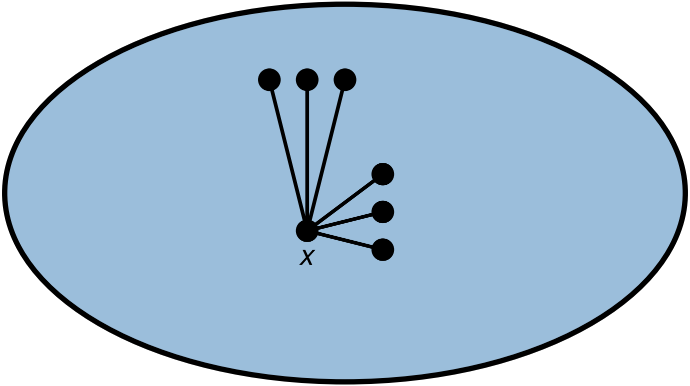
$X = (V, E)$ regular graph of degree k
$k = \lambda_1 \geq \cdots \geq \lambda_n$ adjacency eigenvalues
$\xi = \max\{|\lambda_i| : i \geq 2\}$ zero-weight spectral radius
$S \subseteq V$
$d(S) =$ average degree of subgraph induced by S

Expander Mixing Lemma (Alon, Chung 1988)

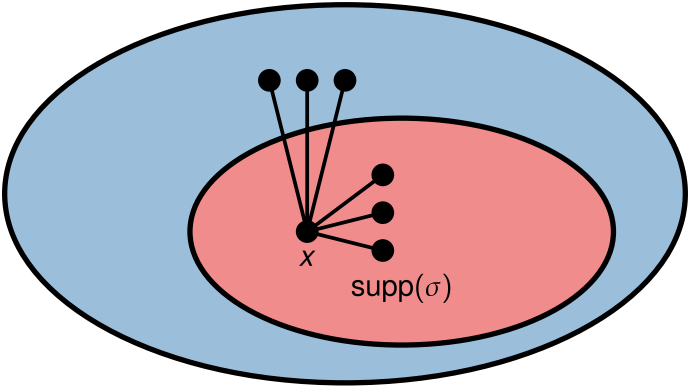
If ξ small then average degree in subset $\approx$ proportional to size of subset:

$$|d(S) - (|S|/n)k| \leq \xi$$

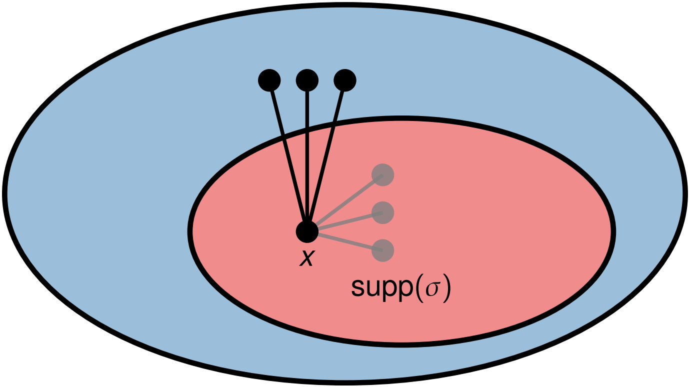
Expander mixing lemma and minimal degree



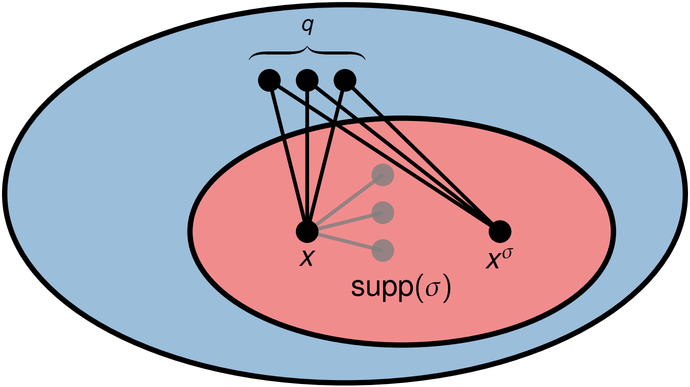
Expander mixing lemma and minimal degree



Expander mixing lemma and minimal degree



Expander mixing lemma and minimal degree



Challenges

- more/better examples of “regularity $\implies$ symmetry” w/o classification
- distance-regular graphs of large diameter and many vertex orbits
- extend the **Sun–Wilmes** combinatorial structure theory of PCCs
- $|\mathrm{Aut}(PCC)| \le$ quasipoly WNE
- or at least same for SR graphs
- extend the **Kivva theorem** to bounded rank: min deg Aut(rank-4 PCC) $\ge \Omega(n)$
- $|\mathrm{Aut}($ proj plane $)| < n^C$
- Steiner t-design $\implies$ $n > C^t$
  (Keevash 2014: $\forall t$)($\exists$ Steiner t-design)
  would imply $|\mathrm{Aut}(X)| \le n^{O(\log n)}$ for Steiner t-designs
  (B–Wilmes)