WL50 6 July 2018

Coherent Configurations and Graph Isomorphism:

The emergence of the Johnson graphs

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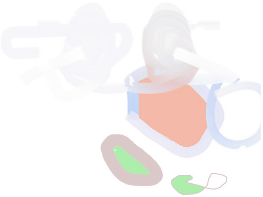

Graph isomorphism

testing isomorphism of graphs with n vertices

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quasipolynomial $\exp((\log n)^c)$ B 2015+

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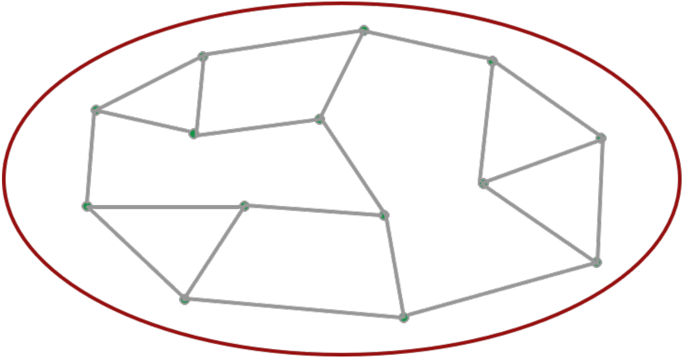
$q(n)$ – **multiplicative cost** (branching factor)

cost analysis: $t(n) \leq q(n)t(0.9n)$

$$\to t(n) \leq q(n)^{O(\log n)}$$

suffices to keep $q(n)$ quasipolynomial

Exposing irregularity: canonical coloring



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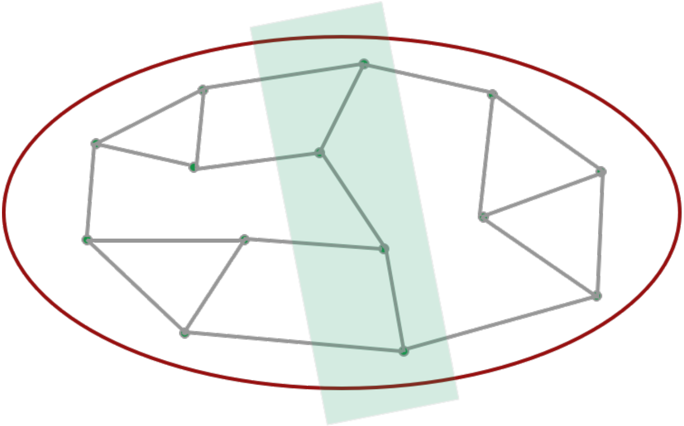

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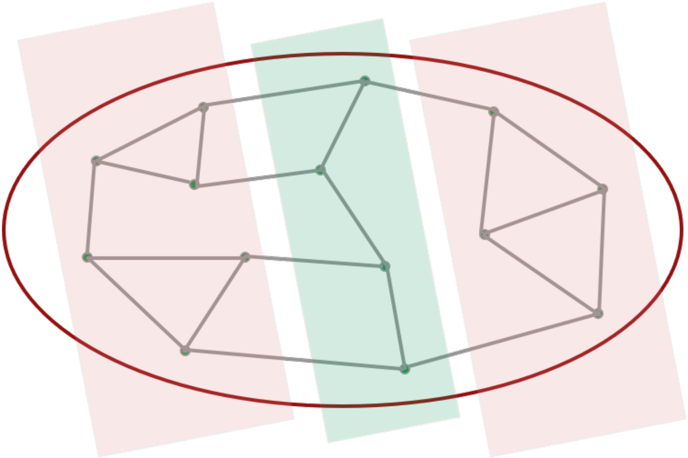

Exposing irregularity: canonical coloring

Good coloring: no dominant color (90% of vertices)

Good equipartition: equipartition of dominant color
nontrivial partition into equal parts

Canonicity: preserved under isomorphisms

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$\rightarrow$ canonical coloring by size of block
each color equipartitioned

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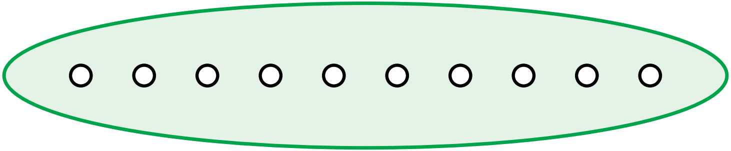

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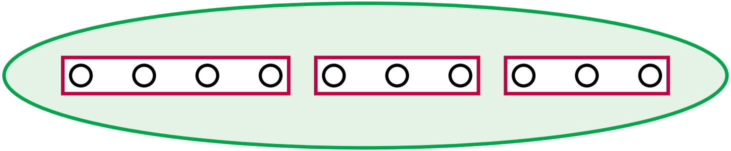

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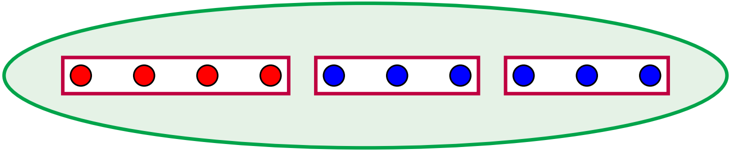

Canonical assignment

Assignment $\mathbf{x} \mapsto F(\mathbf{x})$ structures

E.g. $\mathbf{x}$ – graph, $F(\mathbf{x})$ – coloring of vertices

F **canonical** if it also assigns
isomorphism $\mapsto$ isomorphism

$$\begin{tikzcd}
\mathbf{x} \arrow[r, "\sigma"] \arrow[d, Rightarrow] & \mathbf{y} \arrow[d, Rightarrow] \\
F(\mathbf{x}) & F(\mathbf{y})
\end{tikzcd}$$

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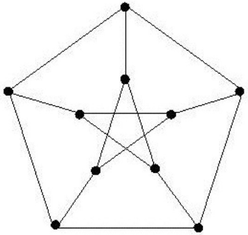
$$\begin{tikzcd} \mathbf{x} \arrow[r, "\sigma"] \arrow[d, Rightarrow] & \mathbf{y} \arrow[d, Rightarrow] \\ F(\mathbf{x}) \arrow[r, "F(\sigma)"'] & F(\mathbf{y}) \end{tikzcd}$$

FUNCTOR between categories of isomorphisms

$$F(\sigma\tau) = F(\sigma)F(\tau)$$

e.g., $\quad F : \text{Graphs} \to \text{ColoredSets}$

Individualization/refinement: creating irregularity



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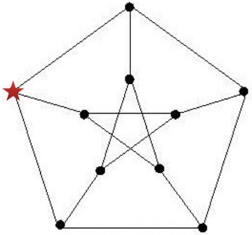
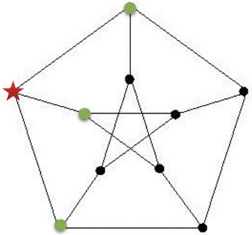


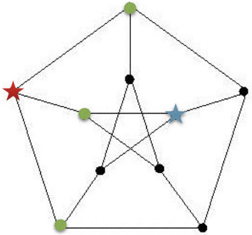
individualize vertex

Individualization/refinement: creating irregularity



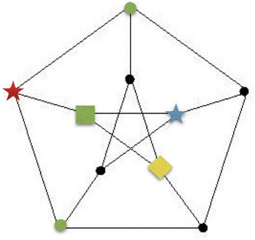
individualize vertex, refine

Individualization/refinement: creating irregularity



individualize vertex, refine
individualize second vertex

Individualization/refinement: creating irregularity



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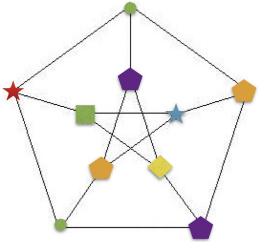

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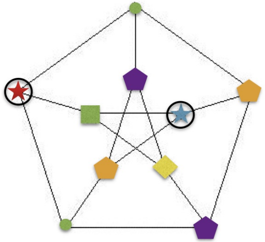

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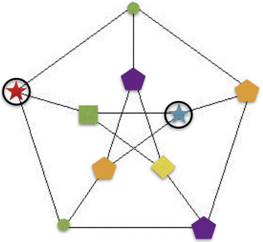

individualize vertex, refine
individualize second vertex, refine
multiplicative cost of individualization
of k vertices: $\mathbf{n^k}$

Cost of good partitioning

Given a nontrivial regular graph
can we find a canonical **good coloring** or **equipartition**
at **modest multiplicative cost?**

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Given a nontrivial regular graph
can we find a canonical **good coloring** or **equipartition**
at **modest multiplicative cost?**

NO! **Johnson graphs**
resilient to good coloring/partition

DEF: $J(k,t)$ **Johnson graph** $t \geq 1 \quad k \geq 2t+1$

vertex set $V = \{v_T \mid T \subseteq \Delta, |T| = t\}$ where $|\Delta| = k$

$$|V| = \binom{k}{t}$$

adjacency: $v_T \sim v_S \iff |T \setminus S| = 1$

---

multiplicative cost of good coloring/partition $\exp(\Omega(k/t))$

important case: $t = 2$ so $n = \Theta(k^2)$ cost $\exp(\Omega(\sqrt{n}))$
Graph Isomorphism **bottleneck** for three decades

Split-or-Johnson

Johnson graphs are the *only* obstruction
to good partitioning

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to good partitioning

Theorem (Split-or-Johnson (abridged))

Given a nontrivial regular graph, at quasipolynomial multiplicative cost one can efficiently find either

(a) *a good canonical vertex-coloring, or*

(b) *a good canonical equipartition, or*

(c) *a canonically embedded Johnson graph on dominant vertex-color class ($\geq 90\%$ of vertices)*

quasipolynomial: $\exp(O((\log n)^c))$

canonicity: functor: isomorphism of input graphs induces isomorphism of embedded structures

efficiently: in quasipolynomial time

Split-or-Johnson, inefficiently

What if we **omit efficiency requirement?**
$\to$ **assume** $\mathrm{Aut}(X)$ **known**

split into orbits; if no orbit dominant, done
let $G := \mathrm{Aut}(X)$ restricted to dominant orbit C, $|C| = m$
$G \neq A_m, S_m$ by symmetry defect
if G t-trans $\implies$ $t = O(\log^2 n / \log\log n)$ (Bochert 1892)
$t = O(\log n)$ (Wielandt 1934)
individualize $t-1$ points $\implies$ G trans, not doubly
if G imprimitive, pick max system of imprimitivity
(minimal domains of imprimitivity)
$\to$ not canonical, but there are $\leq n-1$ of these
b/c blocks containing $x \in \Omega$ are disjoint outside x
$\to$ individualize one of these systems of imprimitivity
remaining case: G **uniprimitive**

Split-or-Johnson, inefficiently

remaining case: G **uniprimitive**
$\rightarrow$ small or Cameron (CFSG)
if small, individualize polylog points $\rightarrow$ fix all
else, if Cameron then socle $(A_r)^s$
if $s \geq 2$ individualize s objects to get imprimitive
else, if $s = 1 \implies$ Johnson

Cameron's classification of large primitive groups
has been used for decades to identify "Luks bottleneck"
for graph isomorphism testing
$\rightarrow$ can be replaced by Split-or-Johnson
$\rightarrow$ eliminates one application of CFSG used to justify quasipolynomial ISO test
other appl of CFSG was eliminated by Pyber

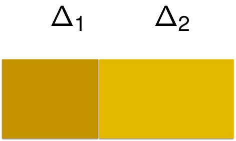
Twins, symmetry defect

$\mathfrak{X} = (\Gamma, \mathcal{R})$ — structure

DEF: $x \neq y \in \Gamma$ **twins** if transposition $(x, y) \in \mathrm{Aut}(\mathfrak{X})$
Fact: "twin or equal" — equivalence relation

DEF: $\Delta \subseteq \Gamma$ **set of twins**: subset of equivalence class
Fact: $\Delta \subseteq \Gamma$ **set of twins** $\iff$ $\mathrm{Sym}(\Delta) \leq \mathrm{Aut}(\mathfrak{X})$
DEF: Symmetricity of $\mathfrak{X}$:
relative size of largest twin equivalence class
DEF: Symmetry defect of $\mathfrak{X}$: $1-$ symmetricity$(\mathfrak{X})$

Example: if $\mathrm{Aut}(\mathfrak{X}) = \mathrm{Sym}(\Delta_1) \times \mathrm{Sym}(\Delta_2)$ where $\Gamma = \Delta_1 \cup \Delta_2$ then the defect of $\mathfrak{X}$ is $\min\{|\Delta_1|, |\Delta_2|\}$



Split-or-Johnson

Exercise. If X is a nontrivial regular graph (not empty, not complete) then defect$(X) \geq 1/2$.

Split-or-Johnson

Exercise. If X is a nontrivial regular graph (not empty, not complete) then $\mathrm{defect}(X) \geq 1/2$.

Theorem (Split-or-Johnson (unabridged))

Given a graph X with $\mathrm{defect}(X) \geq 0.1$, at quasipolynomial multiplicative cost one can efficiently find either

(a) *a good canonical vertex-coloring, or*

(b) *a good canonical equipartition, or*

(c) *a canonically embedded Johnson graph on dominant vertex-color class ($\geq 90\%$ of vertices)*

Why this is the real thing?
Symmetry defect condition obviously **necessary**

Reducing k-ary to binary: Design Lemma

k-ary relation on V: $R \subseteq V^k$
k-ary relational structure: $\mathfrak{X} = (V, \mathcal{R})$ where
$\mathcal{R} = (R_1, \dots, R_m)$ — k-ary relations

Theorem (Design Lemma)

Given a k-ary relational structure $\mathfrak{X}$ with $\mathrm{defect}(\mathfrak{X}) \geq 0.1$, *one can individualize $k-1$ vertices and find, in $n^{O(k)}$ time,*

(a) *a good canonical vertex-coloring, or*
(b) *a good canonical equipartition, or*
(c) *a canonically embedded regular graph on dominant vertex-color class ($\geq 90\%$ of vertices)*

Application to Graph Isomorphism: $k = O(\log n)$

Significance to Graph Isomorphism

Graph Isomorphism can be reduced * via group theory to

Encasement problem

Given a k-ary relational structure $\mathfrak{X} = (V, \mathcal{R})$ with $k = O(\log n)$ (where $n = |V|$) and defect$(\mathfrak{X}) \geq 0.1$, find $G \leq \mathrm{Sym}(V)$ (subgroup of the symmetric group) and $A \subseteq V$ such that

- $|A| = \mathrm{polylog}(n)$
- $\mathrm{Aut}(\mathfrak{X})_{(A)} \leq G$ (pointwise stabilizer of A)
- $|\mathrm{Sym}(V) : G| = \exp(\Omega(n))$

Design lemma & Split-or-Johnson solve this in quasipolynomial time.

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* This reduction took 30 years to find; Split-or-Johnson, 3 weeks

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good coloring: $|S_n : S_k \times S_{n-k}| = \binom{n}{k} > (n/k)^k > 10^{n/10}$

good partition: even better; but what if **Johnson found?**

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$$|\mathrm{Aut}(J(k,t))| = k! \approx \exp(n^{1/t}) \leq \exp(\sqrt{n})$$

– dramatic reduction, can only be repeated $O(\log \log n)$ times

Naive refinement

Input vertex-colored graph

Output refined coloring

$c(x)$ — color of vertex x $\qquad$ I – set of colors

$d_i(x) := \#$ neighbors of x of color $i \in I$

refinement step

for $x \in V$ **do simultaneously**

$$c(x) \leftarrow (c(x); d_i(x) \mid i \in I)$$

Naive refinement

refinement step

for $x \in V$ **do simultaneously**

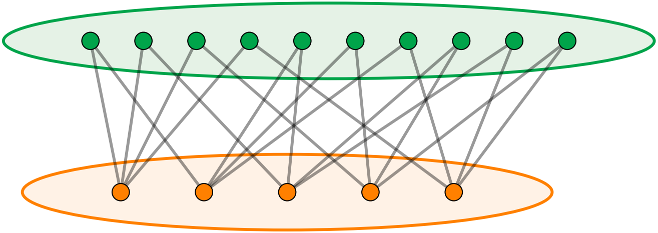
$$c(x) \leftarrow (c(x); d_i(x) \mid i \in I)$$

Repeat until coloring stable

Exercise stable iff **"equitable partition,"** i.e.,

(i) each color class induces regular subgraph

(ii) each pair of color classes induces semiregular bipartite subgraph

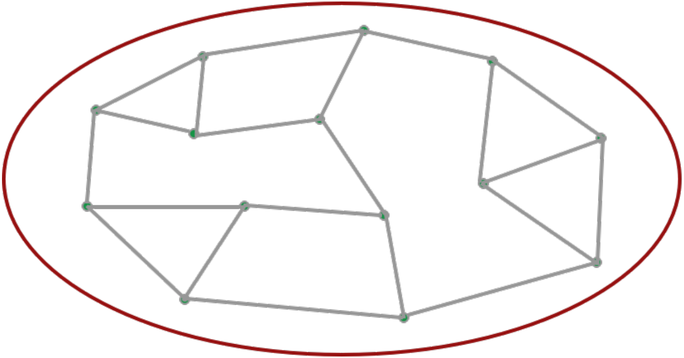


Naive refinement

If input: graph (just one color)

first round: **color** vertices **by degree**

regular graphs: stable, no refinement



Naive refinement

complete split: each vertex gets different color

$\rightarrow$ just 1 candidate isomorphism

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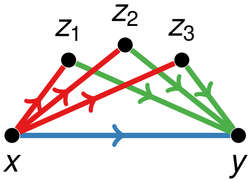
Theorem (B – Erdös – Selkow, 1979)

For almost all graphs, complete split in 2 rounds.
$\therefore$ *ISO test in linear time*

Exercise $\heartsuit$ (Abe Mowshovitz, $\approx$ 1970) If characteristic polynomial of adjacency matrix **irreducible** then
naive refinement $\rightarrow$ complete split.
(Hint: Only equitable partition: discrete)

Advanced refinement: Weisfeiler-Leman 1968

color all ordered pairs, refine by counting triples with shared base and same color composition



coherent configurations: stable under WL

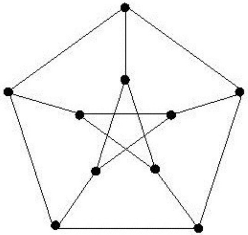
Advanced refinement: Weisfeiler-Leman 1968

Input: graph $\rightarrow$ rank 3 (3 colors):

diagonal $\{(x, x) \mid x \in V\}$,

edge, edge of complement

strongly regular graphs: stable under WL



Coherent configurations

Coherent configuration of rank r: partition of

$V \times V = R_1 \dot\cup \ldots \dot\cup R_r$

R_i — **constituent** (digraph) of color i

coloring: $c(x,y) = i$ if $(x,y) \in R_i$

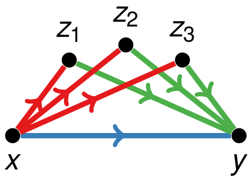
Axioms

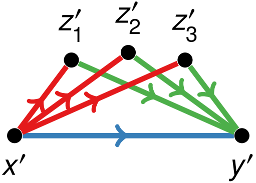
1. diag colors $\neq$ off-diag colors

   $\mathrm{diag}(V) = \{(x,x) \mid x \in V\}$

2. color $x \rightarrow y$ determines color $x \leftarrow y$
3. **intersection numbers:** for any $(x,y) \in R_k$

   p_{ij}^k : $\#\, z$ s.t. $(x,z) \in R_i$ and $(z,y) \in R_j$





Coherent configurations

Homogeneous CC: all vertices same color

Primitive CC: homogeneous and
every off-diagonal constituent (strongly) connected

Schurian case: $G \leq \mathrm{Sym}(\Omega)$
$\mathfrak{X}(G) = (\Omega; \text{orbitals})$ (orbits on $\Omega \times \Omega$)

$\mathfrak{X}(G)$ homogeneous iff G transitive

$\mathfrak{X}(G)$ primitive iff G primitive

$\mathfrak{X}(G)$ uniprimitive iff G primitive and not doubly transitive

Primitice CCs: Johnson schemes

DEF: $\mathfrak{J}(k,t)$ **Johnson scheme** $t \geq 1 \quad k \geq 2t+1$

vertex set $V = \{v_T \mid T \subseteq \Delta, |T| = t\}$ where $|\Delta| = k$

$$|V| = \binom{k}{t}$$

colors: $c(v_T, v_S) = |T \setminus S|$

Example: $\mathfrak{J}(5,2) \quad \sim$ Petersen's graph

Much symmetry: $\mathrm{Aut}(\mathfrak{J}(k,t)) = \mathrm{Aut}(J(k,t)) \cong S_k$

$k! \approx \exp(n^{1/t})$ automorphisms

Coherent configurations

Homogeneous CC: all vertices same color

Primitive CC: homogeneous and
every off-diagonal constituent (strongly) connected

Uniprimitive CC (UPCC):
primitive of rank ≥ 3 (not a clique)

Coherent configurations: a crash course

vertex-color: $c(x) := c(x,x)$

$V = C_1 \cup \cdots \cup C_s$ vertex-color classes

Fact. Edge-color $c(x,y)$ "aware of"
vertex colors $c(x)$ and $c(y)$ $x \to y$

Corollary. $(\forall i)(\exists j,k)(R_i \subseteq C_j \times C_k)$
constituent **homogeneous** $(j = k)$ or **bipartite** $(j \neq k)$

Fact. **Homogeneous** constituent: <u>biregular digraph</u>

Fact. **Bipartite** constituent: <u>semiregular bipartite graph</u>

Coherent configurations: a crash course

Notation. For $R \subseteq V \times V$ and $x \in V$

$R(x) = \{y \in V \mid x \xrightarrow{R} y\}$ — set of **out-neighbors**

Def. **Strong component** of digraph:
equiv class of mutually accessible vertices

Weak component: component of its symmetrization
(ignore orientation)

Fact. The weak components of a *homogeneous constituent* are its strong component.

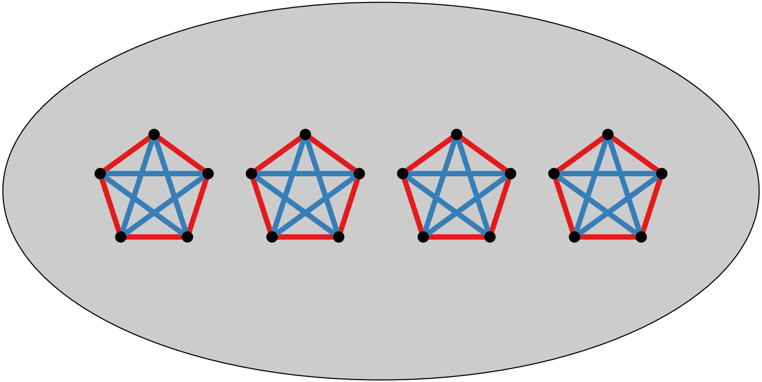
Proof.

DEF: Eulerian digraph: $(\forall x \in V)(\deg^+(x) = \deg^-(x))$

Lemma. Eulerian digraph weakly $\Longrightarrow$ strongly conn

Coherent configurations: a crash course

Fact. The weak components of a constituent have equal order (number of vertices).



Coherent configurations: induced subconfigurations

$X = (V, E)$ digraph — $E \subseteq V \times V$
$\mathfrak{X} = (V, c)$ coherent configuration

Induced subgraph of digraph: for $A \subseteq V$

$$X[A] = (A, E \cap (A \times A))$$

Induced subconfiguration: for $A \subseteq V$

$$\mathfrak{X}[A] = (A, c_{|A \times A})$$

Bipartite graph: $(V_1, V_2; E)$ where $E \subseteq V_1 \times V_2$

Induced bipartite subgraph of digraph:

for $A, B \subseteq V$, $A \cap B = \emptyset$

$$X[A, B] = (A, B; E \cap (A \times B))$$

Induced bipartite subconfiguration:

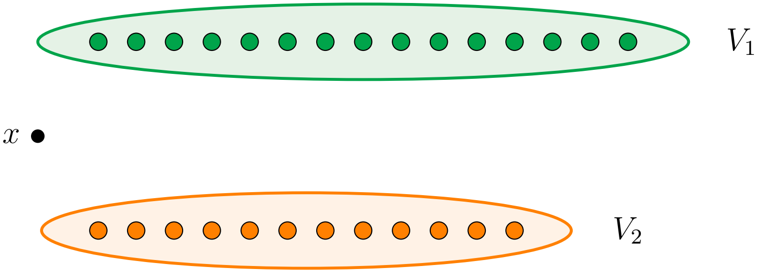
for $A, B \subseteq V$, $A \cap B = \emptyset$

$$\mathfrak{X}[A, B] = (A, B; c_{|A \times B})$$

Fact. $A \subseteq V$ union of color-classes $\implies$ $\mathfrak{X}[A]$ — CC

Coherent configurations: the fix

Exercise. $\mathfrak{X}$ CC, V_1, V_2 vertex-color classes, $x \in V$.
Claim. $R_k[U, W]$ semiregular ($U = R_i(x)$, $W = R_j(x)$)



Coherent configurations: the fix

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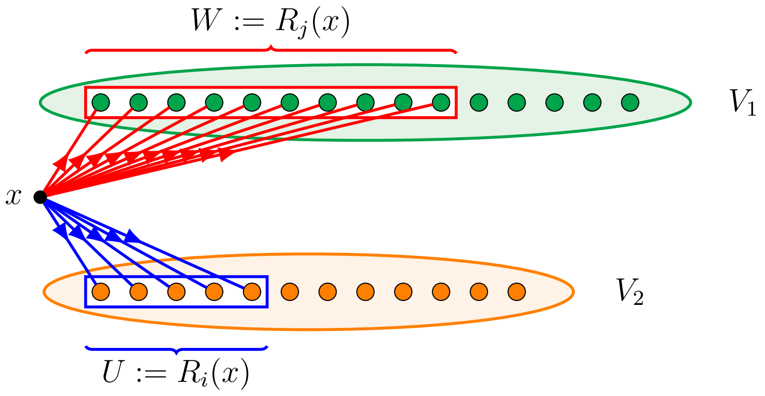

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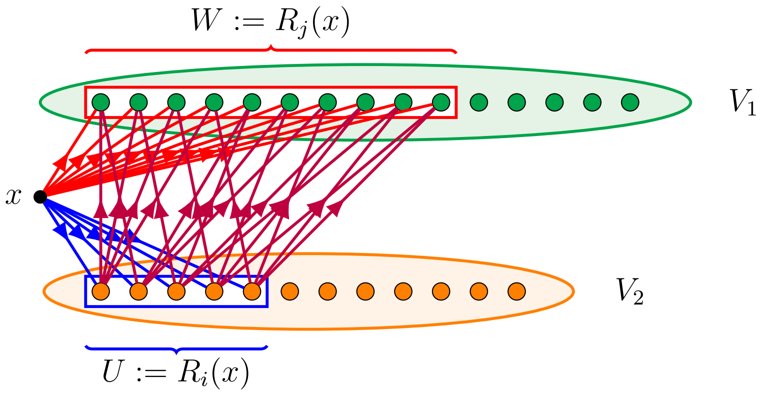

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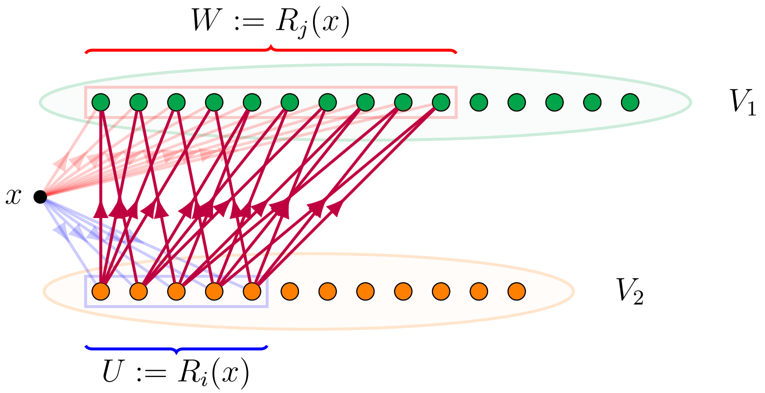

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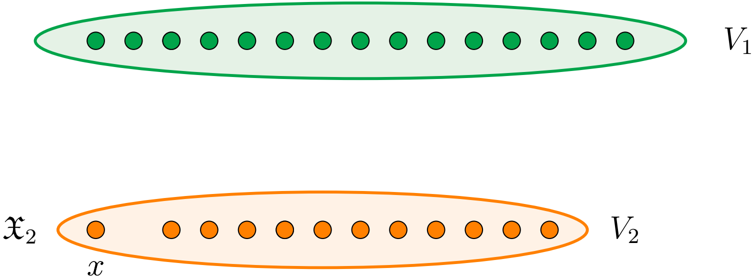
Assume $\mathfrak{X}_2 := \mathfrak{X}[V_2]$ UPCC, $|V_1| > |W| > |V_1|/2$, where $x \in V_2$, $U := R_i(x) \subseteq V_2$, $W = R_j(x) \subseteq V_1$. Then $Y := R_j[U, W]$ nontrivial.



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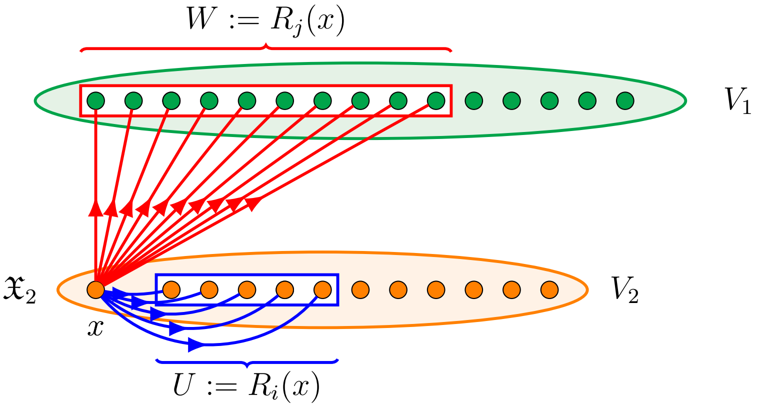

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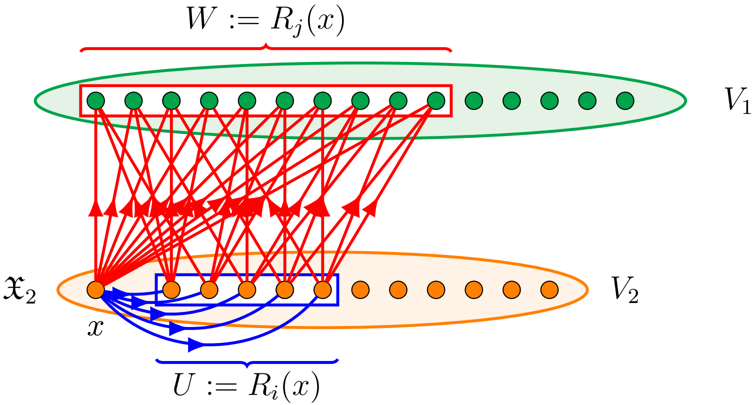

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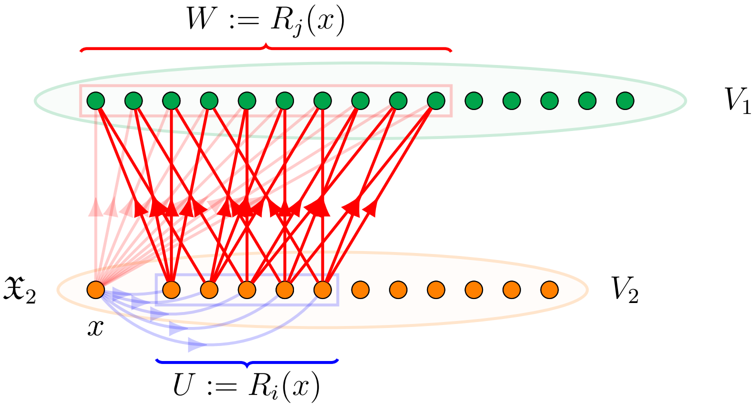
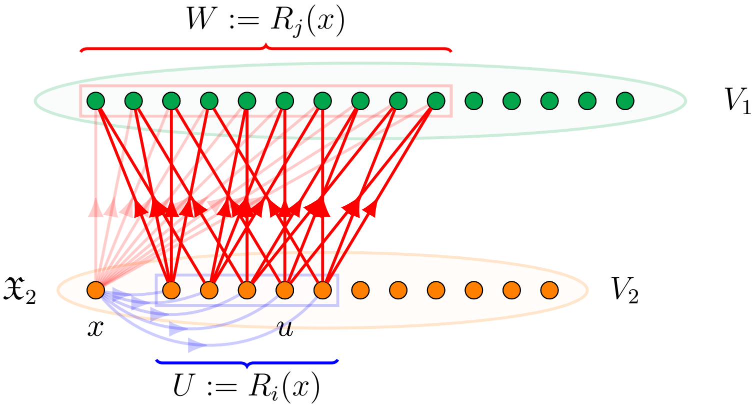


Coherent configurations: the fix

Not empty: $u \in V_2 \implies |R_j(u)| = |R_j(x)| > |V_1|/2$
so $(\exists w \in W)(c(u,w) = j) \quad \therefore (u,v) \in E(Y)$



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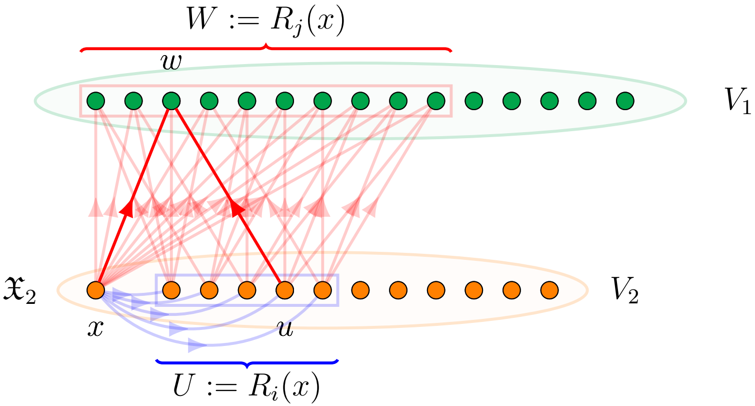

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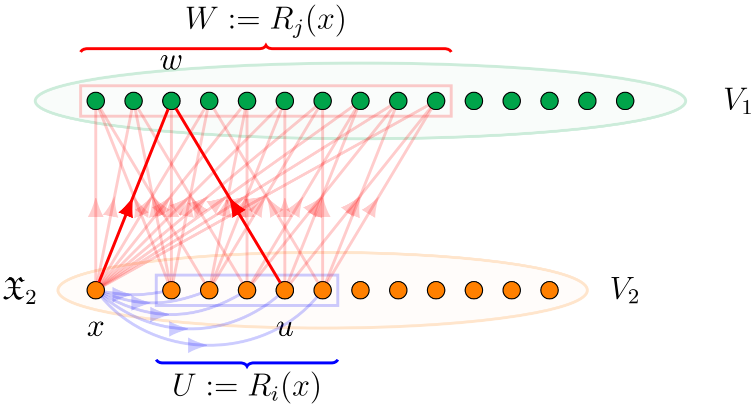
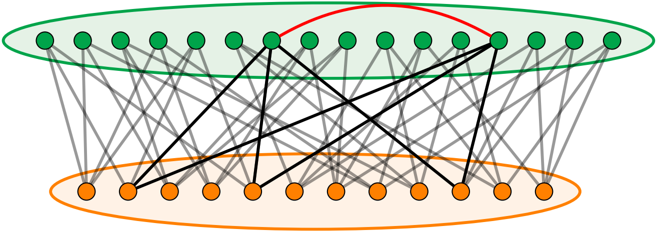


Remains to show: <u>not complete</u>

Coherent configurations: the fix

Lemma (Twin awareness)

A, B vertex-color classes in CC $\mathfrak{X} = (V; R_1, \ldots, R_r)$ and $R_i \subseteq A \times B$. Then for $x \neq y \in A$ the color $c(x, y)$ determines whether or not x, y are twins for R_i



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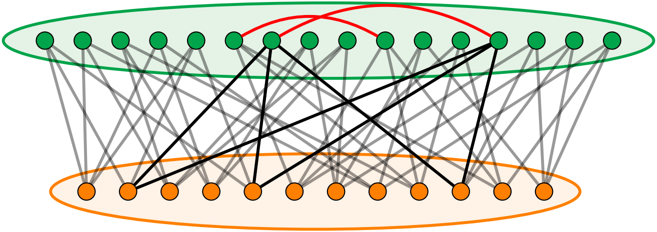

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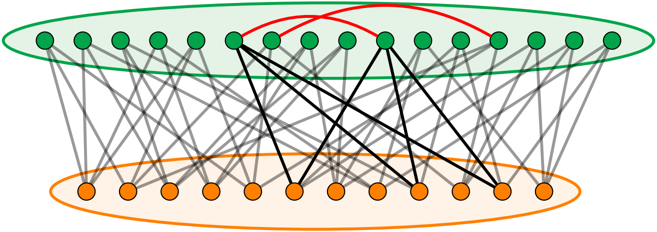
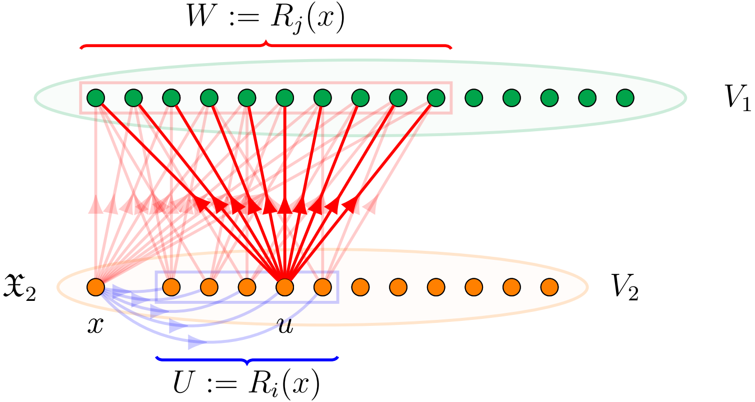


Theorem

Assume $\mathfrak{X}_2$ *UPCC,* $x \in V_2$, $U := R_i(x)$, $W = R_j(x)$, $|W| < |V_1|$. *Then* $Y := R_j[U, W]$ *is not complete.*

Coherent configurations: the fix

If complete $\implies (\forall u \in U)(R_j(u) \supseteq W)$
But $|R_j(u)| = |R_j(x)| = |W| \implies R_j(u) = W$
$\implies x, u$ twins $\implies$ all pairs of color i are twins
but $(V_2; R_i)$ **connected** $\implies$ all vertices in V_2 twins
$\implies (\forall w \in V_1 \setminus W)(deg_i^-(w) = 0) \quad \rightarrow\leftarrow \quad$ QED(Thm)



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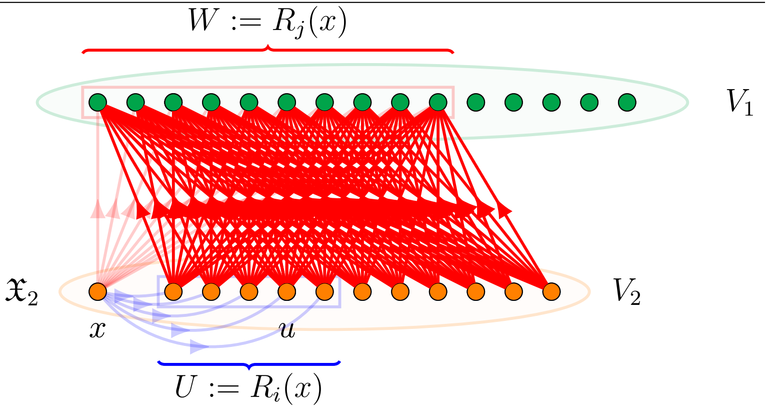


Split-or-Johnson

Dominant vertex-color: more than 90% of vertices
Good coloring: no dominant color
Good equipartition: dominant color nontriv equipartitioned

Theorem (Split-or-Johnson)

Given a graph with $\text{defect}(X) \geq 0.1$*, at quasipolynomial multiplicative cost one can efficiently find either*

(a) *a good canonical vertex-coloring, or*
(b) *a good canonical equipartition, or*
(c) *a canonically embedded Johnson graph on dominant vertex-color class*

canonicity: functor: isomorphism of input graphs induces isomorphism of embedded structures

Reduces to "SoJ for UPCCs" via WL

Split-or-Johnson for UPCCs

Theorem (Split-or-Johnson for UPCCs)

Given a UPCC on vertex-set V, at quasipolynomial multiplicative cost one can efficiently find either

(a) *a good canonical coloring of V, or*

(b) *a good canonical equipartition of V, or*

(c) *a canonically embedded Johnson graph on dominant vertex-color class.*

Reduces to "SoJ for bipartite graphs"

SoJ for semiregular bipartite graphs

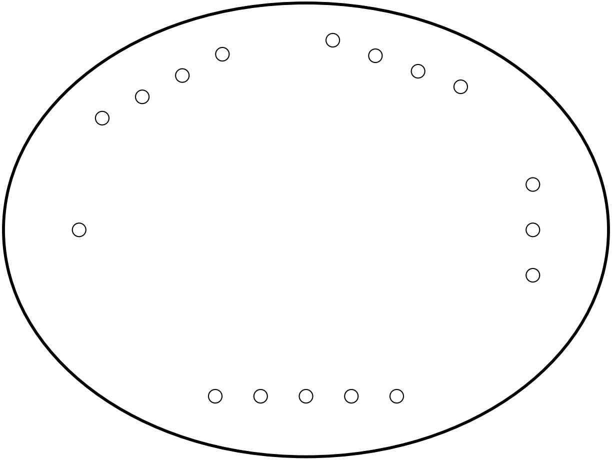
Bipartite graph $(V_1, V_2; E)$ where $E \subseteq V_2 \times V_1$
Notation: $n_i = |V_i|$
Semiregular: all vertices in V_i have same degree $(i = 1, 2)$
Trivial: complete or empty

Theorem (Split-or-Johnson for semireg bipartite graphs)

Given nontrivial semiregular bipartite graph $X = (V_1, V_2; E)$ such that $n_2 \leq 0.9 n_1$, at quasipolynomial multiplicative cost one can efficiently find either

(a) *a good canonical coloring of V_1, or*
(b) *a good canonical equipartition of V_1, or*
(c) *a canonically embedded Johnson graph on dominant vertex-color class of V_1*

Reduction: UPCC-SoJ to semiregular bipartite SoJ



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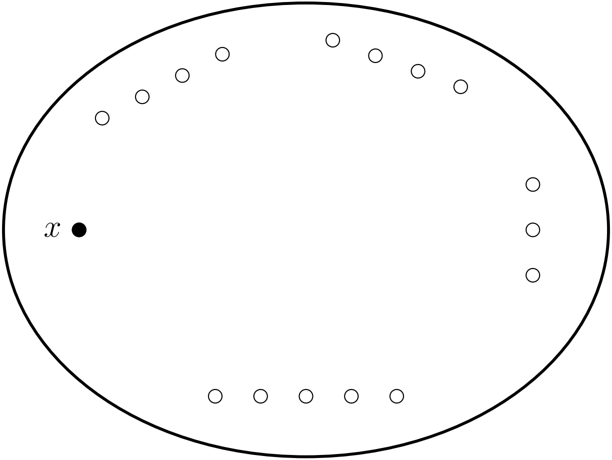

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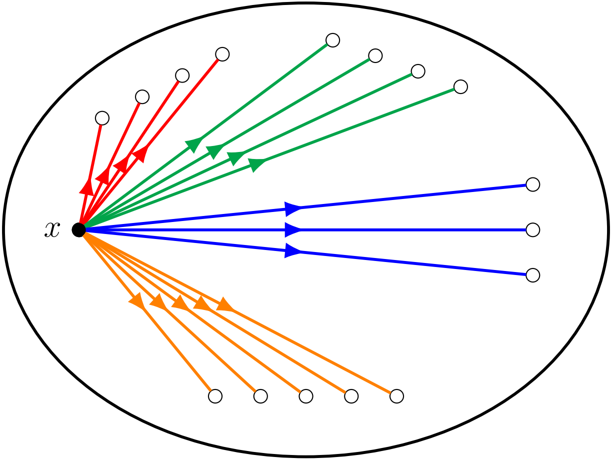

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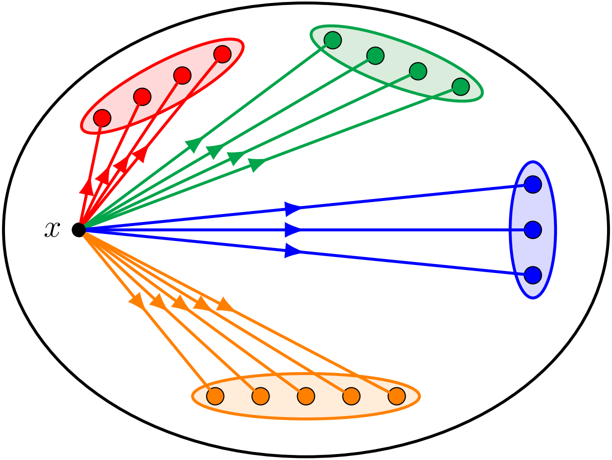

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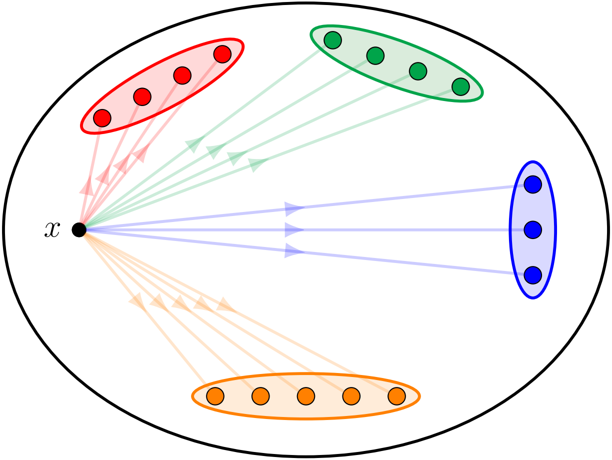

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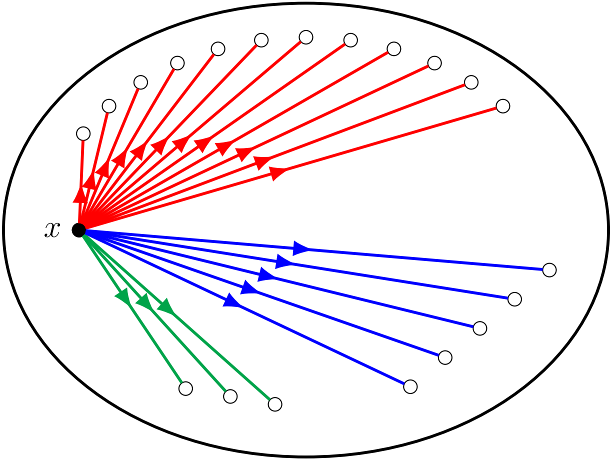

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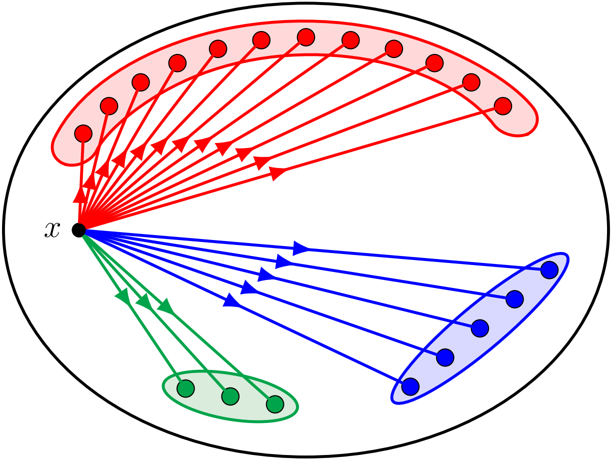

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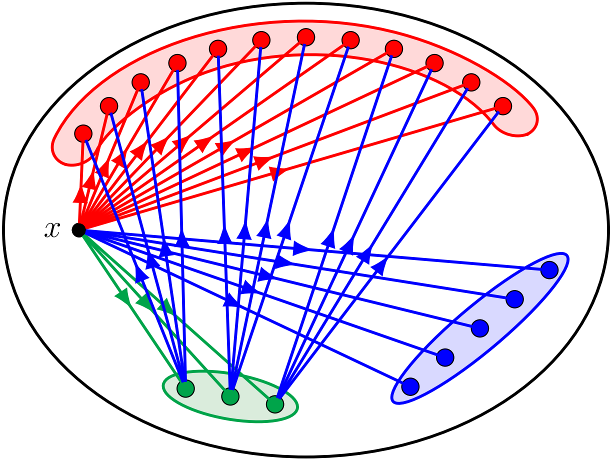

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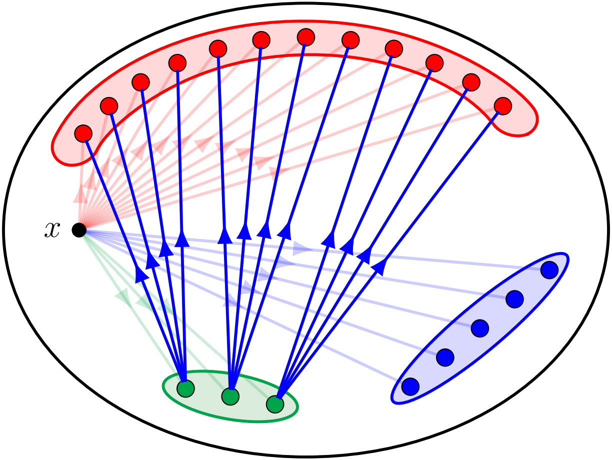


Twins, symmetry defect

Structure $\quad \mathfrak{X} = (V, \mathcal{R})$

Def: $x \neq y \in V$ are **twins** if transposition
$\tau = (x, y) \in \mathrm{Aut}(\mathfrak{X})$

Fact: "twin or equal" — equivalence relation

Def: **Symmetry defect** of $\mathfrak{X}$:

$$\mathrm{defect}(\mathfrak{X}) = 1 - \frac{\max |T|}{|V|}$$ where T — twin equivalence class

Fact: T twin equivalence class $\iff$ $T \subseteq V$ maximal s.t.
$\mathrm{Sym}(T) \leq \mathrm{Aut}(\mathfrak{X})$

Example. If $\mathrm{Aut}(\mathfrak{X}) = S_k \times S_{n-k}$ where $n = |V|$ and $k \leq n/2$ then $\quad \mathrm{defect}(\mathfrak{X}) = k/n$

Twins, symmetry defect for bipartite graphs

Bipartite graph $X = (V_1, V_2; E)$ where $E \subseteq V_1 \times V_2$

$X(v)$ – set of neighbors of v

Fact: $x \neq y \in V_i$ twins $\iff X(x) = X(y)$

Def: Symmetry defect of X in part V_1:

$$\mathrm{defect}_1(X) = 1 - \frac{\max |T|}{|V_1|} \text{ where } T \subseteq V_1 \text{ — twin equiv class}$$

Exercise: If X nontrivial (not empty or complete) semiregular bipartite graph then $\mathrm{defect}_1(X) \geq 1/2$

SoJ for bipartite graphs: unabridged

Theorem (Split-or-Johnson for bipartite graphs)

Given a bipartite graph $X = (V_1, V_2; E)$ with symmetry defect ≥ 0.1 such that $n_2 \leq 0.9n_1$, then at quasipolynomial multiplicative cost one can efficiently find either

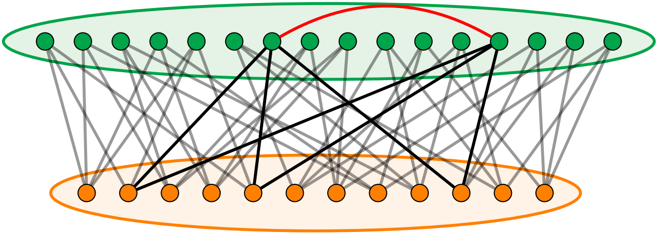
(a) *a good canonical coloring of V_1, or*

(b) *a good canonical equipartition of V_1, or*

(c) *a canonically embedded Johnson graph on dominant vertex-color class of V_1*

Why this is the real thing?

Symmetry defect condition obviously **necessary**



Proof of SoJ for bipartite graphs

Input: bipartite graph $X = (V_1, V_2; E)$ where $n_2 \le 0.9 n_1$,
defect$_X(V_1) \ge 0.1$

Goal: good partition of V_1 or
Johnson graph on $> 90\%$ of V_1

Inductive goal: either achieve "goal"
or halve the size of V_2

May assume defect$_1(X) \ge 0.9$ – otherwise
twin equiv classes — good partition

Start: apply Weisfeiler–Leman to X, obtain CC $\mathfrak{X}$.

$\mathfrak{X}_1 := \mathfrak{X}[V_1], \quad \mathfrak{X}_2 = \mathfrak{X}[V_2]$: CCs

Bipartite part of $\mathfrak{X}$: $\mathfrak{X}_{21} := \mathfrak{X}[V_2, V_1] = \{R_i \mid R_i \subseteq V_2 \times V_1\}$.

Proof of SoJ for bipartite graphs

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Bipartite part of $\mathfrak{X}$: $\mathfrak{X}_{21} := \mathfrak{X}[V_2, V_1] = \{R_i \mid R_i \subseteq V_2 \times V_1\}$.

If $\mathfrak{X}_1$ has no dominant color – goal achieved, **exit**

Otherwise, one can reduce to $\mathfrak{X} \leftarrow \mathfrak{X}[U, W]$ where
$W \subseteq V_1$ dominant color class, $U \subseteq V_2$ color class,
$\mathfrak{X}_{21}$ not monochromatic

$\therefore \mathfrak{X}_1, \mathfrak{X}_2$ homogeneous

May assume $\mathfrak{X}_1$ primitive
otherwise equipartition $\rightarrow$ goal achieved, **exit**

Update $X \leftarrow$ one of the constituents in $\mathfrak{X}_{21}$

So X is **semiregular** $\therefore \text{defect}_i(X) \geq 1/2$

Proof of SoJ for bipartite graphs

Input: CC $\mathfrak{X}$ on $V_1 \cup V_2$
where $\mathfrak{X}_1$ primitive, $\mathfrak{X}_2$ homogeneous,
$\mathfrak{X}_{21}$ not monochromatic, X one of its constituents

Claim: no X-twins in V_1

Proof. By twin-awareness and primitivity of $\mathfrak{X}_1$
the alternative: V_1 is a twin equivalence class
$\therefore$ X trivial, $\mathfrak{X}_{21}$ monochromatic $\rightarrow\leftarrow$ QED

Cases based on $\mathfrak{X}_2$:

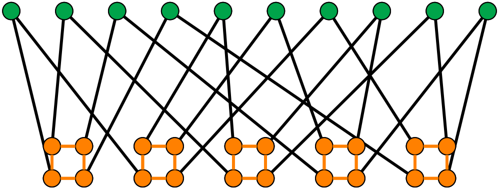
(1) $\mathfrak{X}_2$ imprimitive

(2) $\mathfrak{X}_2$ clique (rank-2 CC)

(3) $\mathfrak{X}_2$ UPCC (uniprimitive coherent configuration)

SoJ for bipartite graphs: imprimitive case

Case (1): $\mathfrak{X}_2$ imprimitive: blocks $V_2 = B_1 \dot\cup \ldots \dot\cup B_k$



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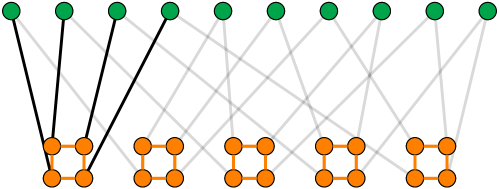
if $\text{defect}_1(X[B_i, V_1]) \geq 1/2$
  return $X \leftarrow X[B_i, V_1]$
  progress: $|B_i| \leq n_2/2$ — V_2 halved



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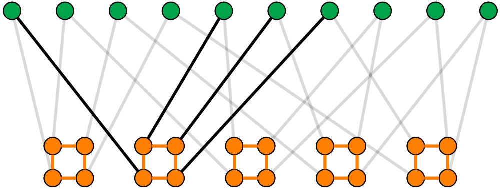

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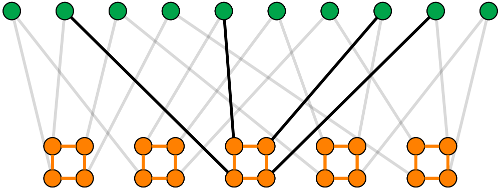

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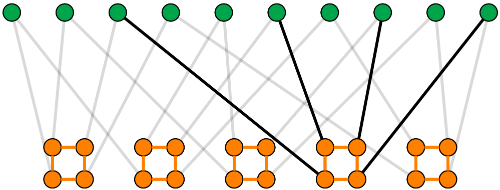

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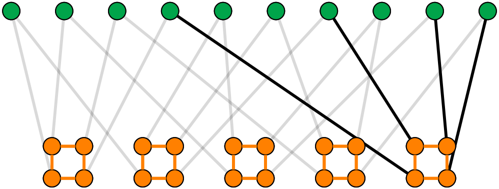

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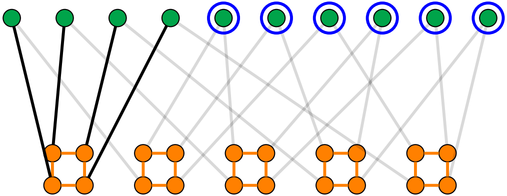

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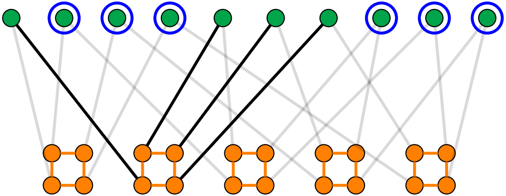

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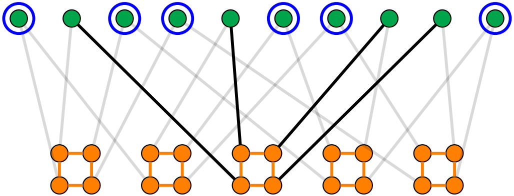

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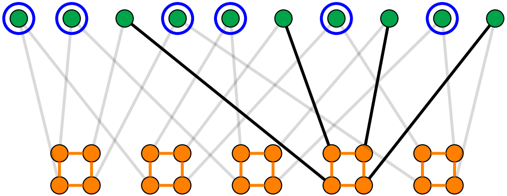
else (:$(\forall i)(\text{defect}_1(X[B_i, V_1]) < 1/2)$:)



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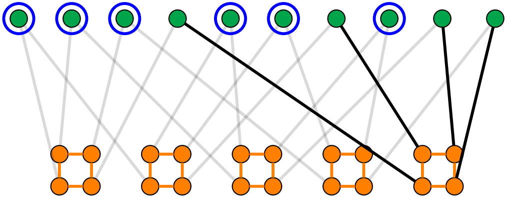

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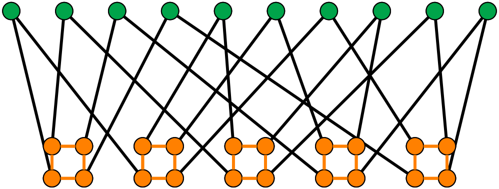

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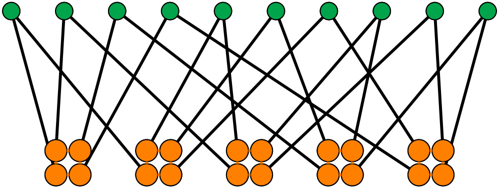

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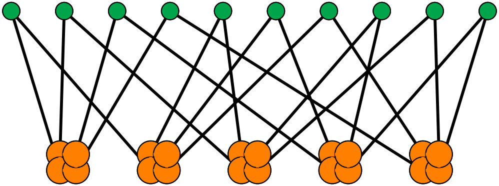

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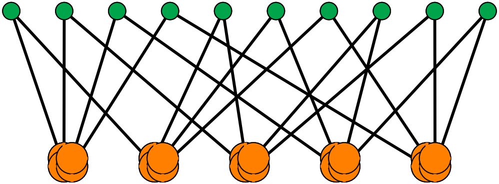

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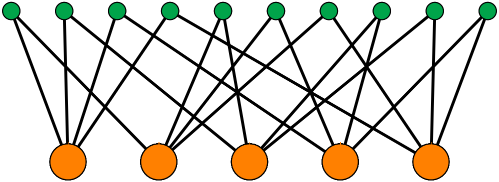


Proof of SoJ for bipartite graphs

Input: CC $\mathfrak{X}$ on $V_1 \cup V_2$
where $\mathfrak{X}_1$ primitive, $\mathfrak{X}_2$ homogeneous,
$\mathfrak{X}_{21}$ not monochromatic, X one of its constituents

Cases based on $\mathfrak{X}_2$:

(1) $\mathfrak{X}_2$ imprimitive ✓

(2) $\mathfrak{X}_2$ clique (rank-2 CC)

(3) $\mathfrak{X}_2$ UPCC (uniprimitive coherent configuration)

SoJ for bipartite graphs: UPCC case

Case (3): $\mathfrak{X}_1$ homogeneous, $\mathfrak{X}_2$ UPCC
$\mathfrak{X}_{21}$ not monochromatic

Recursive goal:
Either find good coloring/equipartition of V_1
or halve V_2

SoJ for bipartite graphs: UPCC case

Case (3): $\mathfrak{X}_1$ homogeneous,$\mathfrak{X}_2$ UPCC
$\mathfrak{X}_{21}$ not monochromatic

Recursive goal:
Either find good coloring/equipartition of V_1
or halve V_2

Algorithm. Individualize $x \in V_2$, refine:
new color of $v \in V_1$; $\quad c'(v) = c(x, v)$

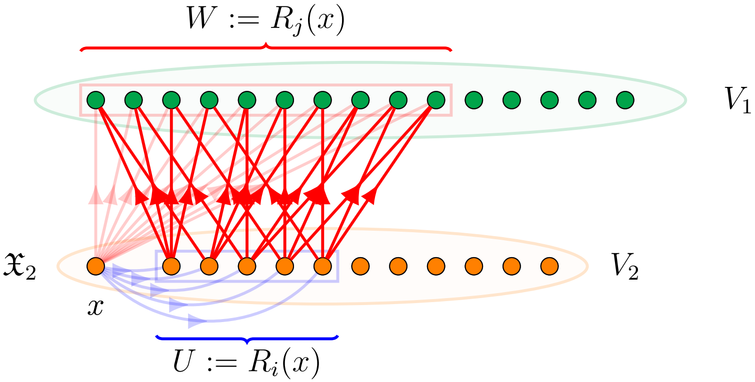
if there is no dominant color in $\mathfrak{X}_{21}$,
this is a good coloring of V_1, goal achieved, **exit**

else: R_j dominant color in $\mathfrak{X}_{21}$

SoJ for bipartite graphs: UPCC case

Situation: R_j dominant color in $\mathfrak{X}_{21}$.
Let $W = R_j(x)$, so $|W| > 0.9n_1$.

Pick a color R_i in $\mathfrak{X}_2$ such that $|R_i(x)| < n_2/2$.
Such color exists b/c $\mathfrak{X}_2$ is not a clique. Let $U = R_i(x)$.

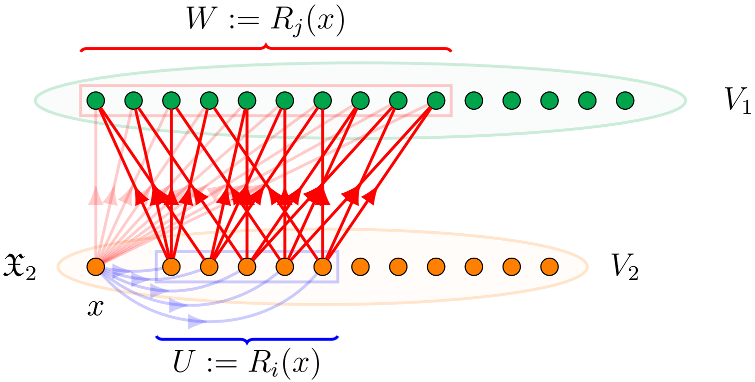


SoJ for bipartite graphs: UPCC case

Let $Y = R_j[U, W]$

return $Y \leftarrow X$. Progress: $|U| < n_2/2$.

Y is semiregular and **nontrivial** by **Theorem**



SoJ for bipartite graphs: UPCC case

Cost analysis

$f(n, m)$ – cost of SoJ for bipartite graph $n = |V_1|$, $m = |V_2|$

reduced $m \leftarrow m/2$ at a cost of 1 individualization

$f(n, m) \leq m \cdot f(n, m/2)\}$

$\therefore \quad f(n, m) = n^{O(\log m)}$ – quasipolynomial QED

Proof of SoJ for bipartite graphs

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Neighborhood hypergraph in bipartite graph

Case (2): $\mathfrak{X}_2$ clique

Desired progress: move to Case (1) or Case (3)
without increasing the n_i

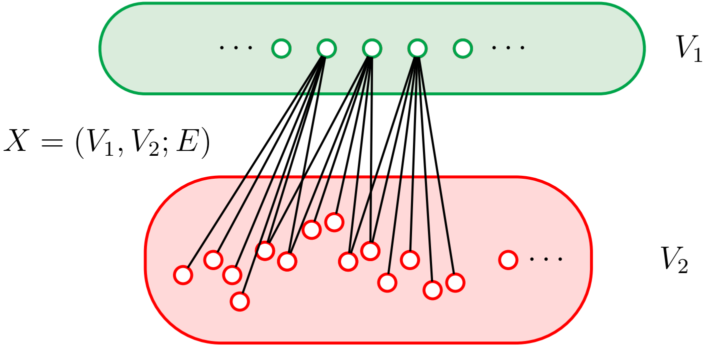
Notation: X graph v vertex $X(v)$: set of neighbors of v

May assume X **semiregular:** $(\forall v \in V_i)(\deg(v) = d_i)$,
nontrivial

Neighborhood hypergraph: $\mathcal{H} = \{X(v) \mid v \in V_1\}$

d_1-uniform d_2-regular hypergraph on V_2

Neighborhood hypergraph in bipartite graph



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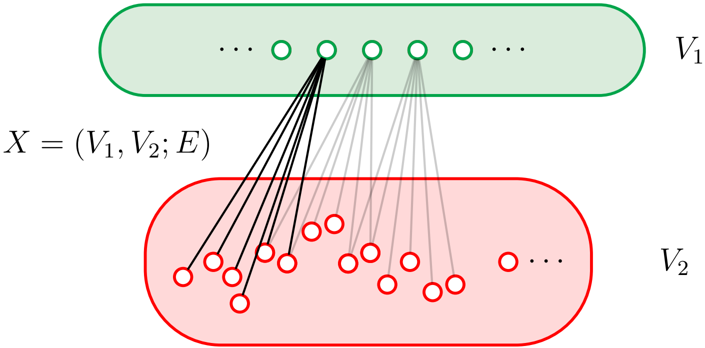

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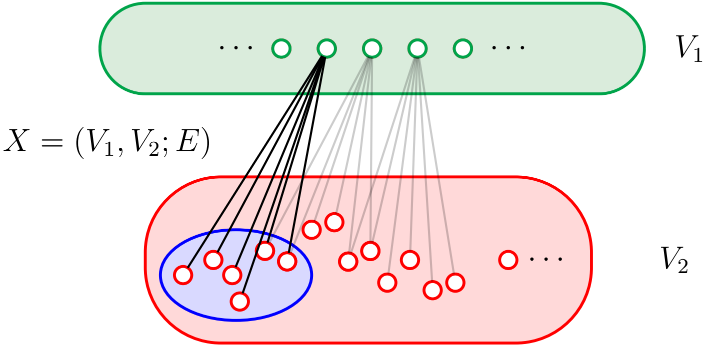

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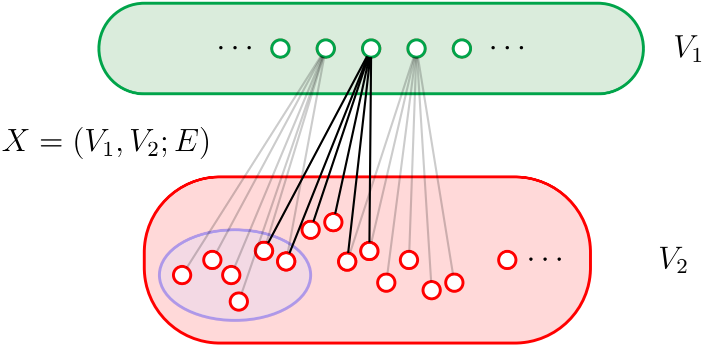

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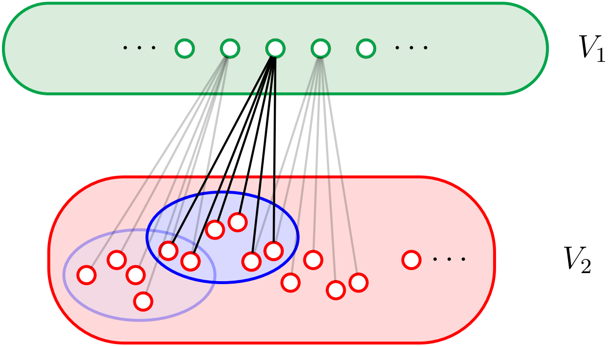

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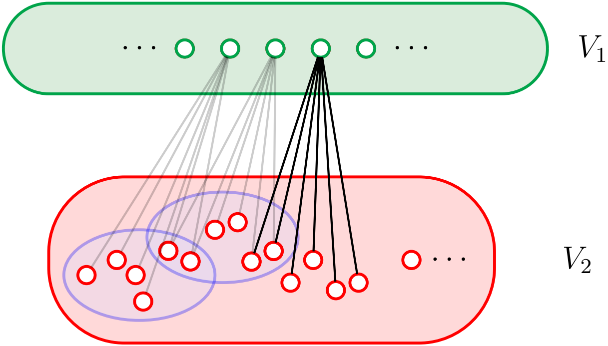

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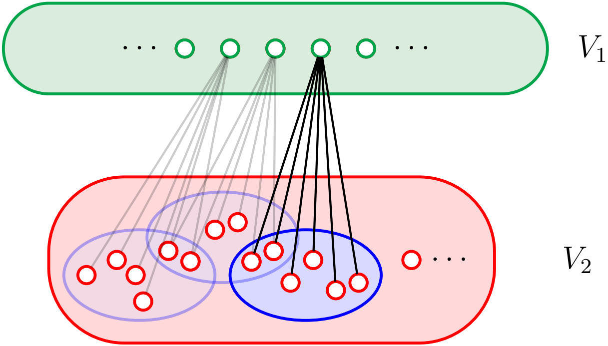

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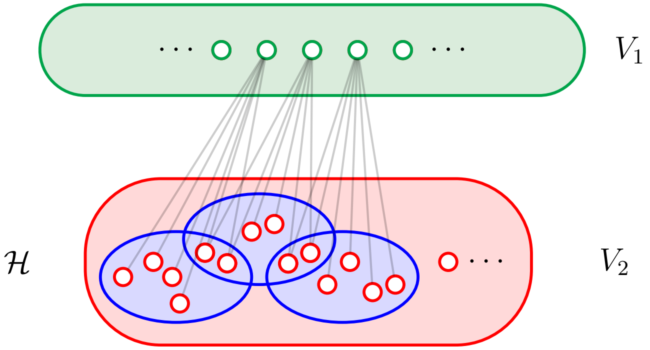

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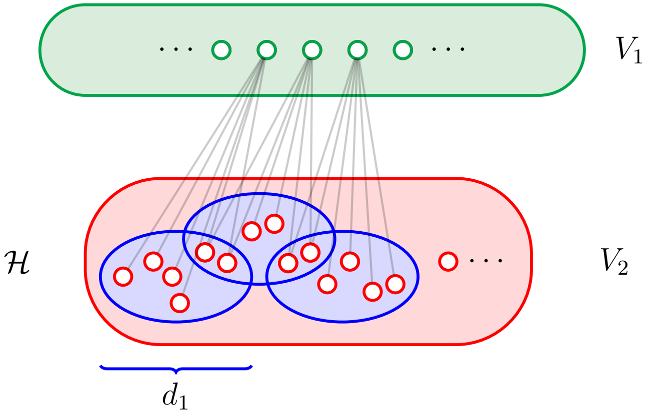

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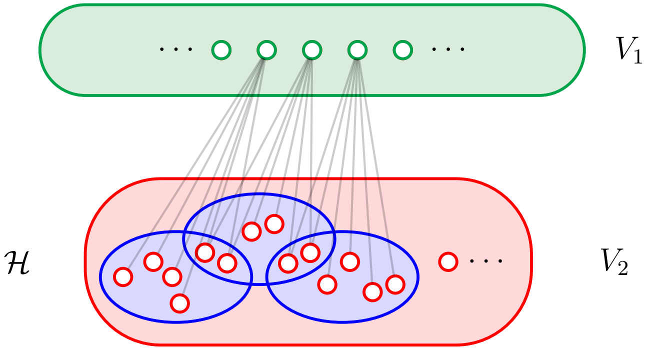

Q: **What if $\mathcal{H}$ complete d_1-uniform hypergraph**?

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$\rightarrow$ **Johnson scheme** $\mathfrak{J}(n_2, d_1)$ canonically on V_1

Canonical: for $x, y \in V_1$ isomorphisms preserve
$|X(x) \cap X(y)|$

This is exactly the case when V_2 is a twin equivalence class for $\mathcal{H}$: $\mathrm{Aut}(\mathcal{H}) = \mathrm{Sym}(V_2)$

Same works for defect$(\mathcal{H}) \leq 1/2$. **Assume** defect$(\mathcal{H}) > 1/2$.

If $d_1 \leq (7/3)\log_2 n_1$: apply **Design Lemma** to $\mathcal{H}$

Neighborhood hypergraph in bipartite graph

If $d_1 > (7/3)\log_2 n_1$: let $t = (7/4)\log_2 n_1$

Def. t-skeleton $\mathcal{H}^{(t)}$ of hypergraph $\mathcal{H}$: all t-subsets of the edges of $\mathcal{H}$.

Lemma (Skeleton defect lemma)

Let $\mathcal{H}$ be a nontrivial, regular, d-uniform hypergraph with n vertices and m edges where $d \le n/2$. Let $(7/4)\log_2 m \le t \le (3/4)d$. Then

$$\text{defect}(\mathcal{H}^{(t)}) \ge 1/4$$

Now apply the Design Lemma to $\mathcal{H}^{(t)}$.

k-ary coherent configurations: definition

DEF: k-ary **partition structure:**

$$\mathfrak{X} = (V, c) \text{ where } \qquad c : V^k \to \{\text{colors}\}$$

DEF: k-ary **configuration:**

- color $c(x_1, \dots, x_k)$ determines equiv relation on $[k]$
$$i \sim j \iff x_i = x_j$$
- $(\forall f : [k] \to [k])$
  color $c(x_1, \dots, x_k)$ determines $c(x_{f(1)}, \dots, x_{f(k)})$

DEF: k-ary **coherent configuration:**

$(\forall i_1, \dots, i_k \le k)(c(x_1, \dots, x_k)$ determines
$(\#z \in V)(\forall j \le k)(c(x_1, \dots, z_{j\text{-th position}}, \dots, x_k) = i_j\,)$
$(r^{k+1}$ **intersection numbers** where $r = |\{\text{colors}\}|)$

k-ary coherent configurations: restriction

DEF: k-ary **partition structure:**

$\mathfrak{X} = (V, c)$ where $\quad c : V^k \to \{\text{colors}\}$

DEF: $(k-\ell)$-ary **restriction** of

k-ary partition structure $\mathfrak{X} = (V, c)$:

let $\vec{x} = (x_1, \ldots, x_\ell) \in V^\ell$, $\; V' = V \setminus \{x_1, \ldots, x_\ell\}$

$\mathfrak{X}_{\vec{x}} := (V', c')$ where for $\vec{y} \in (V')^{k-\ell}$ we set

$$c'(\vec{y}) = c(\vec{x}\vec{y})$$

Fact: Restriction of CC_k is $CC_{k-\ell}$

k-ary coherent configurations: skeleton

DEF: k-ary **partition structure:**

$\mathfrak{X} = (V, c)$ where $\quad c : V^k \to \{\text{colors}\}$

DEF: ℓ-ary **skeleton** of

k-ary partition structure $\mathfrak{X} = (V, c)$:

ℓ-ary partition structure $\mathfrak{X}^{(\ell)} = (V, c')$ where

$$c'(x_1, \ldots, x_\ell) = c(x_1, \ldots, x_\ell, \underbrace{x_\ell, \ldots, x_\ell}_{(k-\ell)\text{ times}})$$

Fact: ℓ-skeleton on CC_k is CC_ℓ

Design Lemma

Theorem (Design Lemma)

Given a k-ary relational structure $\mathfrak{X}$ with $\text{defect}(\mathfrak{X}) \geq 0.1$, *one can individualize $k-1$ vertices and find, in $n^{O(k)}$ time,*

(a) *a good canonical vertex-coloring, or*
(b) *a good canonical equipartition, or*
(c) *a canonically embedded regular graph on dominant vertex-color class ($\geq 90\%$ of vertices)*

An application of Fisher's inequality

Lemma (Large clique lemma)

$\mathfrak{X} = (V, c)$ (classical) coherent configuration with vertex-color classes $W_1, \ldots, W_s$. Assume

- *W_1 dominant:* $(\forall i \geq 2)(|W_1| > |W_i|)$
- *W_1 induces a clique in $\mathfrak{X}$*

Then W_1 is a twin equivalence class in $\mathfrak{X}$.

Proof uses Fisher's inequality for block design:

in a BIBD, $\quad \#$ of blocks $\geq \#$ of vertices

Design Lemma: idea of proof

Let $\mathfrak{X}$ be k-ary coherent, with defect ≥ 0.1.

If 2-skeleton $\mathfrak{X}^{(2)}$ has no dominant color class
DONE (good coloring)

Assume **dominant vertex-color** $C \subseteq V$ $(|C| > 0.9n)$

If $\mathfrak{X}^{(2)}[C]$ **imprimitive**
DONE (good equipartition)

If $\mathfrak{X}^{(2)}[C]$ **uniprimitive** (primitive, not clique)
DONE (reduced to binary case $\rightarrow$ SoJ)

If $\mathfrak{X}^{(2)}[C]$ **clique**
need to destroy it by individualizing $(k-1)$ points

Design Lemma: idea of proof

Case: $\mathfrak{X}^{(2)}[C]$ is a **clique**

How do we **guess** the $\leq (k-1)$ points to **individualize?**

Design Lemma: idea of proof

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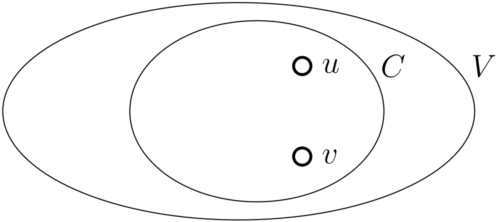

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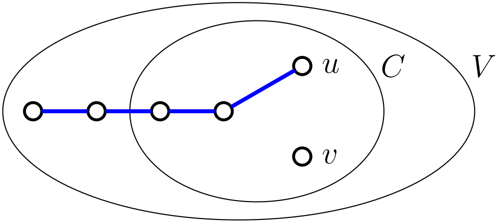

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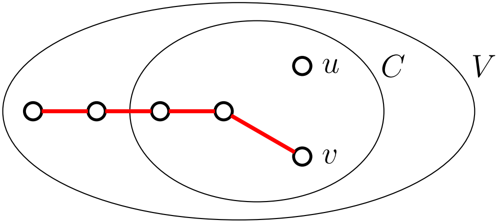

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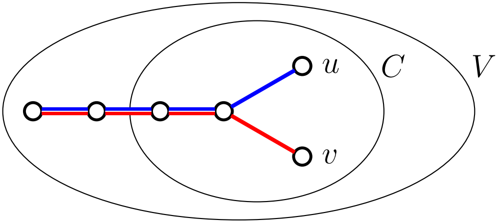

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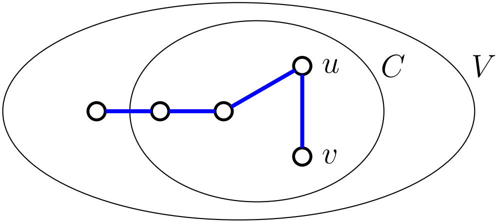

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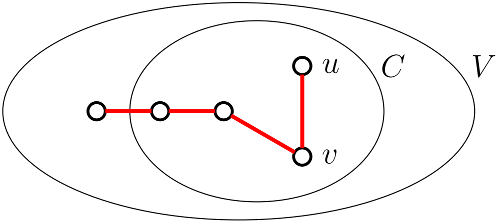

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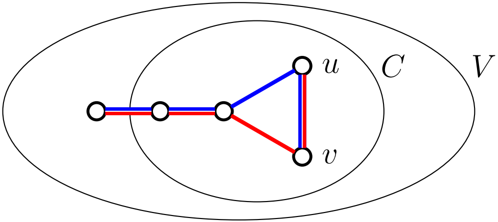
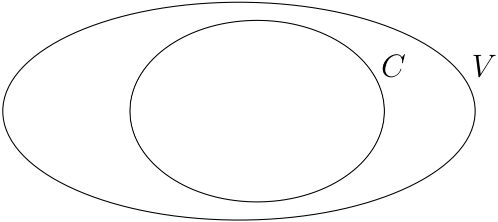


Design Lemma: idea of proof

$\vec{y}$: individualized prefix of $\vec{x}$
Need to destroy clique on $C \setminus \mathrm{supp}(\vec{y})$



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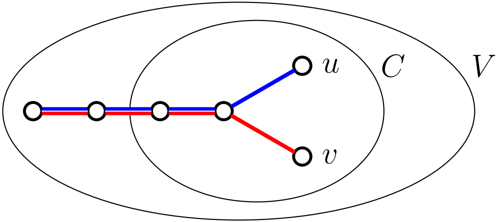

$c(u) \neq c(v)$ after individualizing tail

$\vec{y} = (y_1, \ldots, y_{k-1})$

$\therefore C \setminus \{y_1, \ldots, y_{k-1}\}$ not homogeneous in $\mathfrak{X}_{\vec{y}}$

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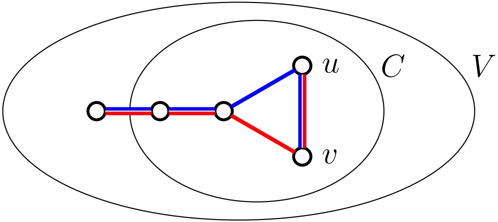

$c(u, v) \neq c(v, u)$ after individualizing tail

$\vec{y} = (y_1, \ldots, y_{k-2})$

$\therefore C \setminus \{y_1, \ldots, y_{k-2}\}$ not a clique in $\mathfrak{X}_{\vec{y}}^{(2)}$

The moral: Hidden irregularity or . . . ?

Split-or-Johnson dichotomy rephrased:

opposite of **hidden irregularity** is

not “hidden regularity” but

The moral: Hidden irregularity or . . . ?

Split-or-Johnson dichotomy rephrased:

opposite of **hidden irregularity** is

not “hidden regularity” but

hidden **robust symmetry**

symmetry that can only be destroyed at high cost