How group theory and statistics met in association schemes

R. A. Bailey

University of St Andrews

QMUL (emerita)

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Abstract I

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R. C. Bose and his collaborators and students generalized earlier work of F. Yates by introducing partially balanced incomplete-block designs for parameters where no balanced incomplete-block design exists. The condition of partial balance ensures that the relevant matrices can be easily inverted by hand, which was important for data analysis in the pre-computer age. This condition relies on the existence of a (symmetric) association scheme.

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Meanwhile, other collaborators of Bose's, including C. R. Nair and J. N. Srivastava, were generalizing association schemes in different ways that now fit within the framework of coherent configurations.

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It is a shame that some of the people that I mention died before all the connections were understood and acknowledged.

Schur rings

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(i) Δ_0 consists just of the identity element of G,

(ii) if Δ_i is one of the subsets then so is $\{g^{-1} : g \in \Delta_i\}$,

(iii) $\chi_i \chi_j$ is a linear combination of $\chi_0, \ldots, \chi_s$ for all i and j,

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Schur (1933) used this idea to prove various results about permutation groups.

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If G has rank r then it has r orbits on $\Omega \times \Omega$.

Suppose that $|\Omega| = n$ and Γ_i is one of these orbits. Its adjacency matrix A_i is the $n \times n$ matrix with rows and columns indexed by Ω and entries

$$A_i(\alpha, \beta) = \begin{cases} 1 & \text{if } (\alpha, \beta) \in \Gamma_i \\ 0 & \text{otherwise.} \end{cases}$$

Chapter V investigates these matrices.

Properties of the adjacency matrices

Let $A_0, \ldots, A_{r-1}$ be the adjacency matrices for the orbits of G on $\Omega \times \Omega$. These satisfy the following conditions.

(i) $A_0 = I$ if G is transitive on Ω;
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G is called generously transitive (Neumann, 1975) if $A_0 = I$ and every A_i is symmetric.

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If A_i is symmetric then the corresponding orbit is self-paired. Its pairs (α, β) can be considered as the edges of an undirected graph Γ on which G acts as a group of automorphisms.
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On the other hand, Higman thought about the whole partition of $\Omega \times \Omega$, starting with groups of rank 3 in 1964, then concentrating on the matrices in 1967.

Pause

I will leave the group theory there for the moment,
and start the story again in experimental design.

Experimental design

Delhi, December 1988:
R. C. Bose Memorial Conference (Statistics)

Słupia Wielka, June 2018:
11th Working Seminar on Statistical Methods in Variety Testing

Blocking

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- within each block, all plots are reasonably alike;
- all blocks contain the same number of plots.

If there are 7 blocks of 8 plots each, then I can grow each variety on one plot per block. This is called a complete-block design.

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I still have 8 varieties and 56 plots,
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3 4 6 8	1 2 5 7	4 5 7 8	1 2 3 6
1 5 6 8	2 3 4 7	2 6 7 8	1 3 4 5
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Balanced incomplete-block designs were introduced by Yates (1936).

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This can easily be done by hand if the design is balanced.

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- If there are 6 blocks of a size 4 then I can use the 6 faces of a $2 \times 2 \times 2$ cube (concurrence is 2 for an edge, 1 for a face-diagonal, 0 for antipode).
- If there are 8 blocks of size 3 then I can develop $\{1, 2, 4\}$ modulo 8
(concurrence is 0 if the difference is ± 4, otherwise is 1).

Square lattice designs

If the number of varieties is m^2 and
there are $g-2$ mutually orthogonal Latin squares of order m,
then a design with gm blocks of size m can be made as follows.

1. Write the varieties in an $m \times m$ square array.
2. The first m blocks are given by the rows;
   the next m blocks are given by the columns.
3. If $g = 2$ then STOP.
4. Otherwise, write down $g-2$ mutually orthogonal Latin squares of order m.
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If $g = m+1$ then the block design is balanced.

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They insisted that each product A_iA_j be a linear combination of $A_0, \ldots, A_s$ in order that the information matrix can be easily inverted by hand.

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(iv) There are integers p_{ij}^k such that if A_i and A_j are adjacency matrices then

$$A_i A_j = \sum_{k=0}^{r-1} p_{ij}^k A_k.$$

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cyclic: the varieties are identified with $\mathbb{Z}_v$ where v is a prime congruent to 1 modulo 4, and the associate class of (i,j) depends on whether or not $i-j$ is a square;

Association schemes of rank 3

Bose and Shimamoto classified association schemes of rank 3 as

group-divisible $\mathrm{GD}(m,n)$: the nm varieties are partitioned into n subsets of size m, and first associates are those in the same subset;

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miscellaneous: they hoped that there were not many more.

Dale Mesner

Dale Mesner was a PhD student at Michican State College (later renamed Michigan State University) from 1950 to 1956, supervised by Leo Katz.
His thesis was called "An investigation of certain combinatorial properties of partially balanced incomplete-block experimental designs and association schemes, with a detailed study of designs of Latin squares and related types".

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His thesis was called “An investigation of certain combinatorial properties of partially balanced incomplete-block experimental designs and association schemes, with a detailed study of designs of Latin squares and related types”.

One important part of this was the development of the algebra generated by the adjacency matrices of an association scheme. He did not know that R. C. Bose had assigned this topic to one of his own PhD students. When Bose heard about Mesner’s work, he suggested collaboration, resulting in the important paper “On linear associative algebras corresponding to association schemes of partially balanced designs” in *Annals of Mathematical Statistics* **30** in 1959.

The Bose–Mesner algebra

The set $\mathcal{A}$ of real linear combinations of the adjacency matrices is called the Bose–Mesner algebra.
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So there are mutually orthogonal subspaces $W_0, \ldots, W_{r-1}$ of $\mathbb{R}^\Omega$ such that if $M \in \mathcal{A}$ then each eigenspace of M is either one of the W_i or the direct sum of two or more of $W_0, \ldots, W_{r-1}$.

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So we have two natural bases for $\mathcal{A}$:
$\{A_0, \ldots, A_{r-1}\}$ is good for doing addition;
$\{Q_0, \ldots, Q_{r-1}\}$ is good for doing multiplication.

Character table and pseudo-inverses

Let λ_{ij} be the eigenvalue of A_i on W_j. Then

$$A_i = \sum_{j=0}^{r-1} \lambda_{ij} Q_j \qquad \text{for } i = 0, \ldots, r-1.$$

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If $M = \theta_1 Q_1 + \cdots + \theta_{r-1} Q_{r-1}$
(with θ_i non-zero for $1 \leq i \leq r-1$)
then the pseudo-inverse of M is $\theta_1^{-1} Q_1 + \cdots + \theta_{r-1}^{-1} Q_{r-1}$.

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- Sets of disjoint lines in $\mathrm{PG}(3, q)$, number 409, in 1964
- On the block structure of certain PBIB designs of partial geometry type, number 457, in 1966.

Negative Latin square type of association scheme

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vertices	valency	triangles per edge
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Mesner's idea was to replace m and g by $-m$ and $-g$ to get an association scheme of negative Latin square type, $\mathrm{NL}_g(m)$.

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$\mathrm{NL}_2(10)$ has 100 vertices, valency 22 and no triangles.

Strongly regular graphs

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The time was ripe to capture the interest of pure combinatorialists and algebraists. Hoffman and Singleton had defined Moore graphs in 1960.

Finite simple groups

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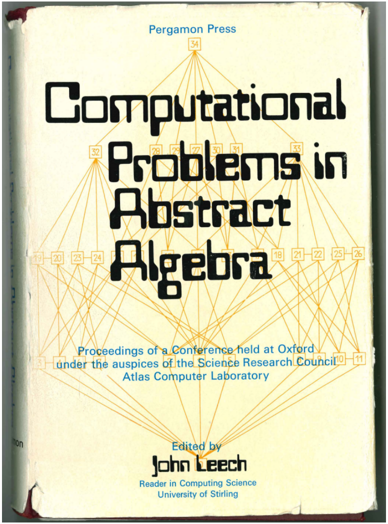
Automorphism groups of highly symmetric combinatorial structures proved a fruitful source.

The non-trivial orbits (on ordered pairs of vertices) of any generously transitive permutation group of rank three are a complementary pair of strongly regular graphs.
D. G. Higman developed an extensive theory of such permutation groups.

1967 conference in Oxford

In 1967 a conference on "Computational Problems in Abstract Algebra" was held in Oxford. At this, Marshall Hall gave a lecture about how the Hall–Janko sporadic simple group had been constructed as (a sub-group of) the automorphism group of a strongly regular graph on 100 vertices with valency 36 and 14 triangles per edge.



Higman–Sims graph and Higman–Sims group

Don Higman and Charles Sims were at this conference, and were inspired by this talk. Sims (Bannai et al., 2009) said that they discussed ideas before, during and after the conference dinner, and by the early hours of the next morning had soon constructed a new sporadic simple group, now called the Higman–Sims group, as a subgroup of index 2 in the automorphism group of a strongly regular graph on 100 vertices with valency 22 and no triangles. They were able to construct this by starting with the Steiner system $\mathfrak{S}(3, 6, 22)$, so they did it with less effort than Mesner.

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If two blocks are disjoint then no block is disjoint from both.

New group; apparently new graph

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Thanks to Edmund Robertson for showing me his article about this which will appear in Mactutor,
and to Colin Campbell for showing me his hard copy of the proceedings of that conference.

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The "bible" on *Distance-Regular Graphs* by Brouwer, Cohen and Neumaier, was published in 1989. It refers to Mesner's 1967 paper, but does not associate him with the Higman-Sims graph.

More details about what Dale Mesner did, and his later life

- T. B. Jajcayová and R. Jajcay: On the contributions of Dale Marsh Mesner. *Bulletin of the Institute of Combinatorics and its Applications* **36** (2002), 46–52.
- T. B. Jajcayová, R. Jajcay and E .S. Kramer: The Life and Mathematics of Dale Marsh Mesner 1923–2009. *Bulletin of the Institute of Combinatorics and its Applications* **59** (2010), 9–30.
- Eiichi Bannai, Robert L. Griess, Jr., Cheryl E. Praeger and Leonard Scott: The Mathematics of Donald Gordon Higman. *Michigan Mathematical Journal* **58** (2009), 3–30.
- Mikhail H. Klin and Andrew J. Woldar: The strongly regular graph with parameters $(100, 22, 0, 6)$: Hidden history and beyond. Dedicated to the memory of Dale Marsh Mesner (1923–2009). *Acta Universitatis Matthiae Belii*, series Mathematics, 2017, 19–76.

Don Higman and coherent configurations

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Higman decided to extend the basic combinatorial ideas to such a set of matrices, irrespective of group actions.
He called this a coherent configuration.
If the diagonal is a single class, then it is a homogeneous coherent configuration.

Don Higman's lecture notes

I was a DPhil student in group theory at Oxford from 1969 to 1972. Don Higman visited for the academic year 1970–1971. He gave a series of lectures on his current thinking on coherent configurations. Peter Cameron and Susannah Howard took notes, which were approved by DGH before being typed up and published as "Combinatorial Considerations about Permutation Groups" in the Mathematical Institute series of lecture notes in 1971.

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Later, DGH developed them into two papers on coherent configurations (1975, 1976).

Did statisticians also go as far as coherent configurations?

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He insisted that if A_i is an adjacency matrix then so is $A_i^\top$; and that the Bose–Mesner algebra is commutative.

Since concurrence matrices of incomplete-block designs are symmetric, this does not change the concept of a partially balanced incomplete-block design.

Multidimensional partial balance

In 1963, Bose and Srivastava started a different generalization, which Srivastava continued for many years.

Suppose that I want to experiment on combinations of v varieties with n quantities of fertilizer.

Let N_{12} be the $v \times n$ matrix whose (i, j)-entry is the number of plots which have variety i with amount j of fertilizer.

Let N_{11} be the $v \times v$ diagonal matrix whose (i, i)-entry is the number of plots with variety i. Define N_{21} and N_{22} similarly.

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The information matrix for varieties is a linear combination of N_{11} and $N_{12}N_{21}$. The information matrix for fertilizer quantities is a linear combination of N_{22} and $N_{21}N_{12}$.

Multidimensional partial balance

In 1963, Bose and Srivastava started a different generalization, which Srivastava continued for many years.

Suppose that I want to experiment on combinations of v varieties with n quantities of fertilizer.

Let N_{12} be the $v \times n$ matrix whose (i, j)-entry is the number of plots which have variety i with amount j of fertilizer.

Let N_{11} be the $v \times v$ diagonal matrix whose (i, i)-entry is the number of plots with variety i. Define N_{21} and N_{22} similarly.

The information matrix for varieties is a linear combination of N_{11} and $N_{12}N_{21}$. The information matrix for fertilizer quantities is a linear combination of N_{22} and $N_{21}N_{12}$.

If these matrices are in the algebra of a coherent configuration with diagonal classes {varieties} and {fertilizer quantities} then the information matrices can be (pseudo-)inverted easily.

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Bose and Srivastava called these multidimensional partial balance schemes. They generalize to 3 or more diagonal classes.

Who knew what?

D. G. Higman died on 13 February 2006.
J. N. Srivastava died on 18 November 2010.
It seems that neither of them knew of each other's work.

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