2-Closure of $\frac{3}{2}$-transitive permutation groups in polynomial time

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$\frac{3}{2}$-Transitive permutation groups

Definitions (Wielandt, 1964)

Permutation group on Ω is $\frac{1}{2}$-transitive, if its orbits on Ω have the same size.

Permutation group on Ω is $\frac{3}{2}$-transitive, if it is transitive, and stabilizer of point α is $\frac{1}{2}$-transitive on $\Omega \setminus \{\alpha\}$.

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Liebeck-Praeger-Saxl, 2015

The classification of $\frac{3}{2}$-transitive permutation groups and $\frac{1}{2}$-transitive linear groups.

2-Closure of permutation groups

Let G be a permutation group on Ω.

Action of G on Ω induces the action on Ω^2: $(\alpha, \beta)^g = (\alpha^g, \beta^g)$.

Denote as $\mathrm{Orb}_2(G)$ the set of orbits of its action (2-orbits).

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2-Closure problem

Given a permutation group G, find a set of generators of $G^{(2)}$.

2-Closure problem can be reduced to the Graph Isomorphism Problem

Some results

Ponomarenko, 1994

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$\mathrm{ASL}_0(\mathbb{F}) = \mathbb{Z}_p^d \rtimes \mathrm{SL}_0(\mathbb{F})$ – small Passman group,
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5: **output** G.