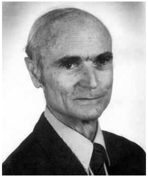
Graphs and Groups

Rob Curtis

Pilsen July 2018

Some students of H.F.Baker (1866-1956)

, , , 

Coxeter (1907-2003), du Val (1903-1987), Edge (1904-1997) and Todd (1908-1994).

Also Semple, Pedoe, Mordell and Bronowski among others

The *synthematic totals* preserved by the symmetric group S_6

$\infty 0.14.23$	$\infty 0.12.34$	$\infty 0.13.42$
$\infty 1.20.34$	$\infty 1.30.42$	$\infty 1.40.23$
$\infty 2.31.40$	$\infty 3.41.20$	$\infty 4.21.30$
$\infty 3.42.01$	$\infty 4.23.01$	$\infty 2.34.01$
$\infty 4.03.12$	$\infty 2.04.13$	$\infty 3.02.14$
$(\infty/01234)$	$(\infty/01342)$	$(\infty/01423)$

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$\infty 1.30.42$	$\infty 1.40.23$	$\infty 1.20.43$
$\infty 3.21.04$	$\infty 4.31.02$	$\infty 2.41.30$
$\infty 2.43.10$	$\infty 3.24.10$	$\infty 4.32.01$
$\infty 4.02.31$	$\infty 2.03.41$	$\infty 3.04.12$
$(\infty/01324)$	$(\infty/01432)$	$(\infty/01243)$

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- Similar lectures were being given at the same time in Oxford by Graham Higman.

, , 

Three Johns: John Leech, John McKay and John Conway

Todd's 1966 paper on M_{24}

A representation of the Mathieu group M_{24}
as a collineation group.

by J. A. Todd (Cambridge, England)

In memory of Guido Castelnuovo, in the recurrence of the first centenary of his birth.

Summary. - *The Mathieu group M_{24} can be represented as a collineation group in space of 11 dimensions over the field of two elements. This paper discusses the geometry associated with this representation.*

Introduction.

The quintuply transitive Mathieu group M_{24}, of degree 24 and order $24.23.22.21.20.48 = 244, 823, 040$, is the automorphism group of a Steiner system $S(5, 8, 24)$, that is, an arrangement of 24 objects in sets of 8, such that any 5 objects belong to exactly one set. It has been shown by Witt [5] that such a system is essentially unique. Some years ago I proved [4] that M_{24} could be represented as a collineation group in a projective space of 11 dimensions over the field $GF(2)$. The object of this paper is to develop the properties of this representation in some detail, and in particular to show how a number of maximal subgroups of M_{24} can be interpreted geometrically in a very simple manner.

In § 1 below we describe the main properties of the Steiner system $S(5, 8, 24)$ which will be used in the sequel. The projective space is introduced in § 2. In § 3 we describe the individual operations of the group, and their invariant subspaces when the group is represented as a collineation group. Finally, in § 4, we describe 8 maximal subgroups of M_{24}. In the ten tables which follow the paper we give, first, an explicit form of the Steiner system $S(5, 8, 24)$ and then the character tables of M_{24} and the 8 maximal subgroups.

§ 1. - **The Steiner system $S(5, 8, 24)$.**

1.1. - The Steiner system $S(5, 8, 24)$ is an arrangement of 24 elements in sets of 8, such that any 5 of the elements belong to exactly one set. We assume Witt's result [5] that such a system is unique to within isomorphism.

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- Fixing an octad O, the octads which intersect O in 4 points form a Steiner system S(3,4,16) on the complementary 16-ad.

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t				
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- These intersect columns with the same parity and rows with the same parity.

Correspondence with partitions of the octad into halves

- There are $(2^4-1)(2^4-2)/(2^2-1)(2^2-2) = 35$ 2-dimensional subspaces of a 4-dimensional space over $\mathbb{Z}_2$, and $\frac{1}{2}\binom{8}{4} = 35$ partitions of an octad into two tetrads.

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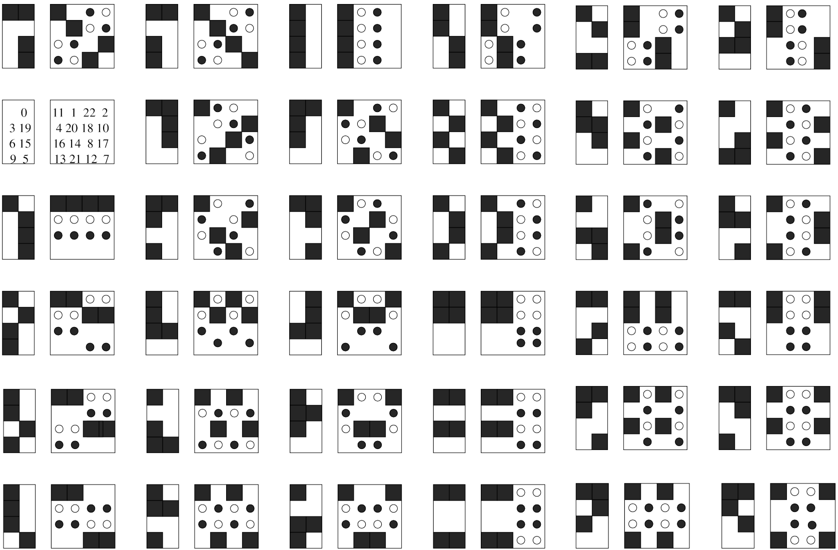
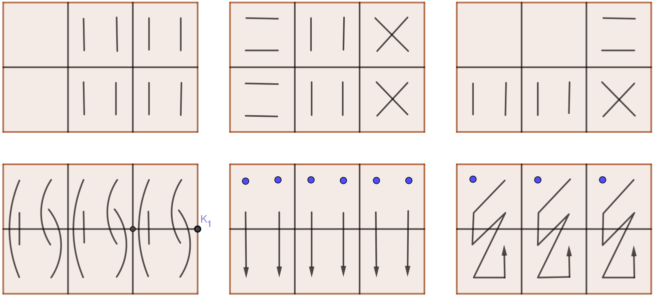
- The resulting 35 pictures exhibit all 759 octads in an instantly recognisable form.

The Miracle Octad Generator or MOG



Some elements of M_{24}



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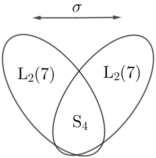

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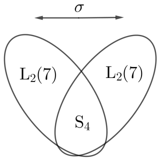

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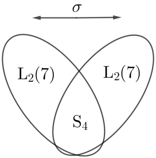

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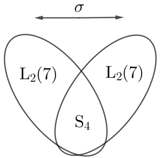

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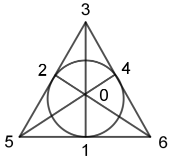
- The 7 images of σ under conjugation by one of the two copies of $\mathrm{L}_2(7)$ must generate M_{24}.
- Given $\mathrm{L}_2(7)$ acting on 24 letters we can immediately write down permutations which generate M_{24}.

A combinatorial interpretation



The Fano plane

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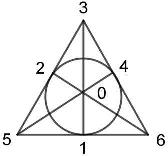

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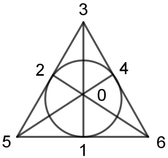

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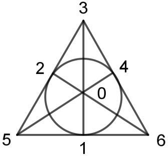

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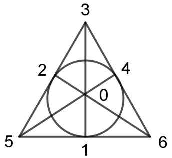

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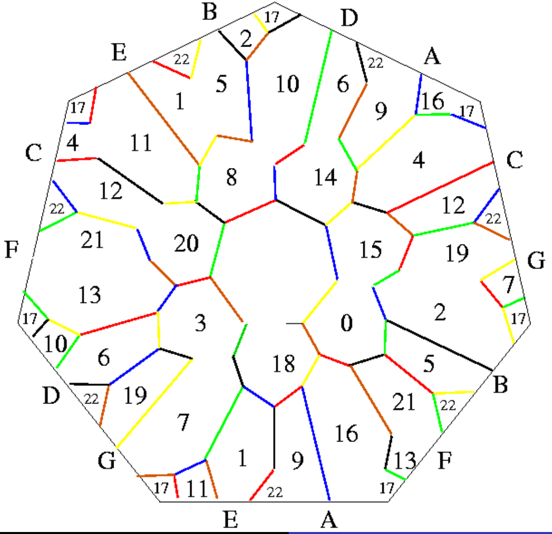
- Then $\langle t_a \mid a \in \{0, 1, \ldots, 6\}\rangle \cong \mathrm{M}_{24}$.

A geometric interpretation: M_{24} acting on the 24 faces of the Klein map



And so to graphs!

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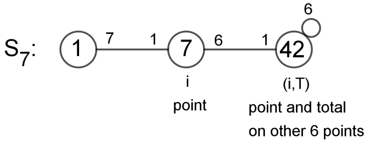
- The Lemma: If $\pi = w(t_1, t_2)$ then $\pi \in \mathrm{C}_N(N_{12})$, the centralizer in N of the stabilizer in N of points 1 and 2.

The Hoffman-Singleton graph



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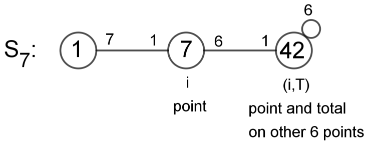
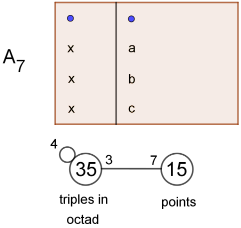


- 

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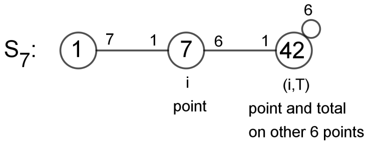

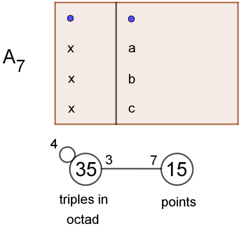

- So form $2^{\star 50} : (\mathrm{U}_3(5) : 2)$ and seek relator by which to factor.

The shortest possible relator

- There are two possible 2-point stabilizers N_{12} depending on whether points 1 and 2 are joined or not: (i) the stabilizer of an edge is S_6 with trivial centralizer; (ii) the stabilizer of a non-edge is S_5 which centralizes an involution.

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- If i —— j —— k is a path of length 3, then $\mathrm{C}_N(N_{ik}) = \langle (i,k) \rangle$, which fixes j and the other 5 points joined to j. Wish to write (i,k) as a word in t_i, t_j and t_k.

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$$\frac{2^{\star 50} : (\mathrm{U}_3(5) : 2)}{(i,k) = t_i t_k t_i t_j} \cong \mathrm{HS} : 2.$$

the Higman-Sims sporadic simple group and its outer AM.

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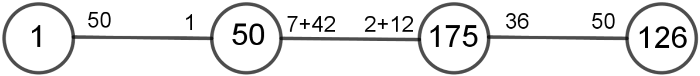

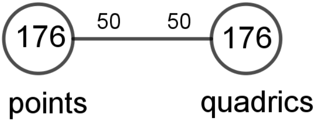

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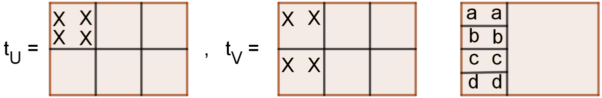

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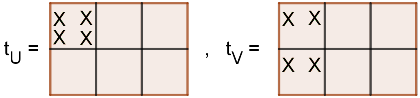

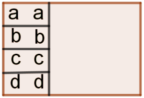

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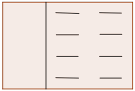
- We find $C_N(N_{UV}) = \langle$ $\nu =$  $\rangle$.

Obtaining the Conway group $\cdot O$

- It turns out that

$$\frac{2^{\star\binom{24}{4}} : \mathrm{M}_{24}}{\nu = t_{ab}t_{ac}t_{ad}} \cong \cdot\mathrm{O}.$$

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- Allowing the group to act on the standard basis vectors produces a copy of the Leech lattice.
- The generators t_U so obtained are essentially the Conway elements ζ_T which he used to show the lattice was preserved by more than just $2^{12} : \mathrm{M}_{24}$.

The action of M_{24} on *trios*.

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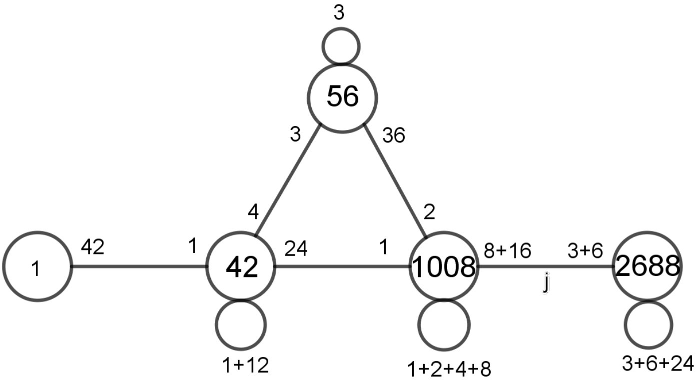

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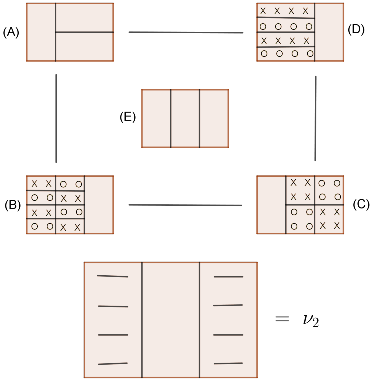
The relation for J_4.



$$= \nu_2$$

▶

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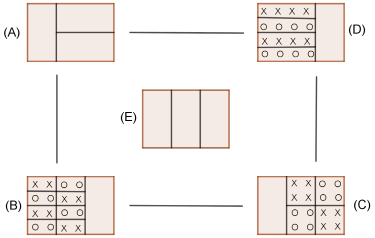

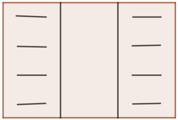

$= \nu_2$

-
- We find that

$$\frac{2^{\star 3795} : M_{24}}{\nu_2 = t_A t_B t_A t_D} \cong J_4 \times 2.$$