Notes to Robert Curtis's presentation at WL2018

R. T. Curtis

Some students of H.F. Baker

Henry Frederick Baker was a hugely influential Cambridge geometer in the late 19th and early 20th century. His students included Coxeter who went on to become one of the most important geometers in the 20C; du Val and Edge, who were distinguished algebraic geometers; and Todd of the Todd-Coxeter coset enumeration algorithm. The eminent number theorist Mordell was another student, as was Jacob Bronowski who wrote and presented *The Ascent of Man* which in the 1960s was a popular television series about the rise of civilization. Bronowski's daughter Lisa (later Lisa Jardine) also studied Mathematics at Cambridge in the year above me but changed to English in her third year and went on to become Professor of Renaissance Studies at Queen Mary, University of London. This talk will begin with the contribution of Todd.

The *synthematic totals* preserved by the symmetric group S_6

The 15 partitions of six letters into pairs are known as *synthemes*; a set of five synthemes such that every pair appears is a *synthematic total*; there are just 6 of these totals, and so the symmetric group S_6 permutes both the original 6 letters and the 6 totals. Todd wrote the totals on the board in his 1967 Cambridge Part III course which I attended, and demonstrated important properties of these two *non-permutation identical* actions.

From S_6 to M_{12} to M_{24}

Todd's lectures demonstrated an important method of constructing groups: Use a well-known group to construct a new combinatorial or geometric structure; observe that the new structure possesses more symmetries than just the group you used in its construction.

The Leech lattice and the Conway group

John Leech, a Cambridge mathematician turned computer scientist who became a professor at the University of Sterling, discovered the lattice named after him in connection with sphere-packing in 24 dimensions. Its construction ensured that it was preserved by the Mathieu group M_{24} together with sign changes on the codewords of the binary Golay code. He believed he knew the order of its group of symmetries to within a factor of 2, but could not prove the three orbits on minimal vectors under this group of shape $2^{12} : \mathrm{M}_{24}$ fused in a larger group. John McKay suggested to John Conway, who at the time was predominantly a number theorist and logician, that he work out its group of symmetries. The result was three new sporadic simple groups.

Todd's 1966 paper on M_{24}

Todd's paper concluded with a list several pages long of all 759 octads of the Steiner system S(5,8,24). This list was used extensively at the time by people investigating the Leech lattice and the Conway groups, but this method was laborious and unsatisfactory. I determined to find a more revealing description of the system.

The Steiner system S(3,4,16)

I first fixed an octad and arranged the complementary 16-ad into a 4×4 array so that the tetrads in which octads intersect it were plainly recognisable.

Correspondence with partitions of the octad into halves

This gave a correspondence between the 35 partitions of the fixed octad into two fours with 35 sets of four tetrads in the complementary 16-ad. The breakthrough was the realisation that the 24 points could be partitioned into three disjoint octads so that this visual correspondence was the same whichever of the three octads we fix.

The Miracle Octad Generator or MOG

The resulting *Miracle Octad Generator* enables one instantly to recognise when an 8-element subset is an octad; to complete any 5-element subset to the unique octad containing it; and to write down elements of M_{24} having desired properties.

Some elements of M_{24}

Many elements of M_{24} have a particularly simple and appealing form in the MOG. Here the top row shows involutions of cycle shape $1^8.2^8$ and 2^{12}; the second row gives generators (of orders 2, 3 and 7 respectively) for the copy of $L_2(7)$ which has the same action in each of the three bricks, and thus commutes with the S_3 bodily interchanging them.

An innocent question

Many years later Tony Gardiner, a colleague of mine at the University of Birmingham, asked me whether two copies of $L_2(7)$ in M_{24} could intersect in a copy of S_4; he wanted to use this to construct a certain distance transitive graph. I found that this was possible using subgroups from the class of maximal $L_2(7)$s, which act transitively on the 24 points, and that when this happens there is an involution σ interchanging the two copies of $L_2(7)$ and commuting with their intersection. It is easy to see that the 7 conjugates of σ under the action of one of the two $L_2(7)$s must generate M_{24}, and moreover one can readily write down these 7 elements given $L_2(7)$ acting on 24 points.

A combinatorial interpretation

For instance one can take a class of 7-cycles in $L_3(2)$ as the set of 24 objects. Then there are 6 different ways in which one can write down these generators of M_{24}, one of which is shown in the slide. These give the six copies of M_{24} which contain our original $L_2(7)$ as a maximal subgroup.

A geometric interpretation: M_{24} acting on the 24 faces of the Klein map

The *Klein quartic* is usually taken to be $x^3y + y^3z + z^3x = 0$. In the associated *Klein map* the 24 faces represent the points of inflexion; the 56 vertices represent the points where the 28 bitangents touch the curve; and the 84 edges represent the *sextactic points*, points at which conics make 6-fold contact. The figure shows a 14-gon whose edges are identified in pairs, so defining a surface of genus 3; thus if we leave on face 16 across boundary edge A in the top right, we re-enter the map still on face 16 at face A on the base of the figure. The curve is preserved by a copy of $L_2(7)$ and so generators of M_{24} will be visible on the map. Explicitly, the 84 edges fall into 7 blocks of imprimitivity of size 12 under the action of $L_2(7)$; we have coloured these with 7 colours. Now choose a colour, red say, and define a permutation of shape 2^{12} which interchanges every pair of faces which have a red edge in common; thus $4 \leftrightarrow 12, 10 \leftrightarrow 14, \ldots$. Note that the central face should be labelled ∞ and so $\infty \leftrightarrow 8$. The 7 involutions corresponding to the 7 colours generate M_{24}. The Mathieu group M_{24} and the Klein map were both discovered in the second half of the 19th century, but there was no connection between them. It is a source of great delight to me that generators of the former are readily observable on the edges of the latter.

And so to graphs!

The notation $m^{\star n}$ denotes a free product of n copies of the cyclic group C_m, thus M_{24} is a homomorphic image of $2^{\star 7} : L_2(7)$. In a similar manner if the group N acts as automorphisms of the graph Γ of degree n, then we can interpret the vertices of the graph as elements of order 2 and form the infinite semi-direct product $2^{\star n} : N$. Every element of this group may be written as πw where $\pi \in N$ and w is a word in the n generators of the free product; thus any relator by which we wish to factor to obtain a finite image has this form. Effectively we are saying how a permutation of N can be written in terms of the 7 *symmetric generators*.

The Lemma tells us which permutations can possibly be written in terms of just two symmetric generators: simply consider the centralizers of the various 2-point stabilizers.

The Hoffman-Singleton graph

The beautiful Hoffman-Singleton graph is preserved by a group isomorphic to $U_3(5) : 2$ with point stabilizer S_7. Fixing one of the 50 vertices, $\star$ say, the remaining points fall into a suborbit of length 7 labelled by $\mathbb{Z}_7$ and a suborbit of length 42 labelled (i, T) where $i \in \mathbb{Z}_7$ and T is a synthematic total on $\mathbb{Z}_7 \setminus i$. The fixed point $\star$ is joined to i; i is joined to (i, T); and (i, T) is joined to $(j, T^{(i,j)})$.

We may also see a copy of the graph in the MOG: The stabilizer of an octad in M_{24} is isomorphic to $2^4 : A_8$; fixing a point x outside the octad is a copy of A_8; and fixing a further point y inside the octad is a group isomorphic to A_7 acting transitively on the remaining 7 points of the octad and the 15 remaining points of the complementary 16-ad. Take as vertices the 35 triples of elements of the 7-orbit together with the 15 orbit. Join two triples if they are disjoint and join a triple to the three points outside the fixed octad which complete the triple together with x and y to an octad.

We now apply the Lemma.

The shortest possible relator

According to the Lemma there is no non-trivial element of $N \cong \mathrm{U}_3(5) : 2$ which can be written in terms of generators t_i and t_j without causing collapse if i and j are joined. If i is not joined to k then there is a unique vertex j joined to both of them. The stabilizer of i and k is thus isomorphic to S_5, the subgroup of the S_7 fixing j which also fixes i and k. Centralizing this 2-point stabilizer is $\langle (i,k) \rangle \cong \mathbb{Z}_2$, interchanging i and k and fixing the other 5 points joined to j. Attempting to write (i,k) in terms of t_i, t_j and t_k, we find that words of length 3 or less cause collapse and that all words of length 4 which do not lead to collapse are equivalent to $(i,k) = t_i t_k t_i t_j$.

Factoring by this relation leads to HS : 2, the automorphism group of the Higman-Sims group.

Graham Higman's geometry

The group is named after Donald Higman and Charles Sims, but this approach produces, through a short manual double coset enumeration, the geometry of Graham Higman which has 176 points, 176 quadrics with 50 points on each quadric and 50 quadrics through each point. The group HS acts doubly transitively on points and quadrics and the outer automorphism interchanges them.

Bigger fish: using M_{24} as *control subgroup*

We now consider M_{24} acting on the $\binom{24}{4}$ tetrads of the 24 points; note that this is *not* a primitive action. As before we apply the Lemma and consider the various 2-point stabilizers and their centralizers. With t_U and t_V as shown we again find that there is a unique nontrivial element that can be written in terms of them, but also note that the 2-point stabilizer fixes a further 4 tetrads. So we may attempt to write this involution ν as a short word in these 6 symmetric generators.

Obtaining the Conway group $\cdot O$

Theoretical considerations show that there is a word of length 3 which does not lead to collapse, and factoring by this relation yields the Conway group ·O.

This approach may be used to obtain matrices generating the Conway group, together with the Leech lattice on which they act.

The action of M_{24} on *trios*.

M_{24} has a primitive action on the 3795 trios, partitions of the 24 points into 3 disjoint octads.This action has rank 5 with suborbits as shown in the diagram and valence 42 if we join two trios which have an octad in common. That is to say, if their intersection matrix is of form $\begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{pmatrix}$.

The relation for J_4.

We concentrate on the 1008-orbit which corresponds to trios having intersection matrix of form $\begin{pmatrix} 0 & 4 & 4 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \end{pmatrix}$ with the fixed trio. A and B in the diagram are so related, and their stabilizer in M_{24} fixes the 3 further trios shown. Applying the Lemma tells us that the only non-trivial element which can be written in terms of these 5 trios is ν_2 as shown.

In fact the shortest word which does not lead to collapse is $\nu_2 = t_A t_B t_A t_D$, and factoring by this relation gives us the largest Janko group direct product with $\mathbb{Z}_2$.

Note that this relation could not have defined the simple group as the infinite group

$$2^{\star 3795} : \mathrm{M}_{24}$$

possesses a subgroup of index 2, namely all elements πw in which w has even length; our additional relator lies in this subgroup.

In order to "kill off" this redundant $\mathbb{Z}_2$ we may factor by a second relation $t_A t_E t_C = 1$. I used both relations in my original work on this application but John Bray of Queen Mary, University of London, produced a neat argument to show that the length 4 relation was (almost) sufficient to define the group!