The Graham Higman method and beyond

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Symmetry vs Regularity
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Symmetry vs Regularity

- Regularity: association schemes
- Symmetry: automorphisms of association schemes
- The Graham Higman "method":

Regularity $\longrightarrow$ ~~Symmetry~~

Algebra

Outline

- Higman's observation
- Survey of its various applications and extensions
- Some perspectives

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Association schemes

Adjacency matrices of a (symmetric) association scheme $\mathcal{S}$:

$$\mathsf{A}_0 (= \mathsf{I}), \mathsf{A}_1, \ldots, \mathsf{A}_D \in \mathbb{R}^{V \times V}$$

Eigenspaces:

$$\mathbb{R}^V = \mathbf{W}_0 \oplus \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_D$$

	$\mathbf{W}_0$	$\mathbf{W}_1$	$\ldots$	$\mathbf{W}_D$
A_0	P_{00}	P_{01}	$\ldots$	P_{0D}
A_1	P_{10}	P_{11}	$\ldots$	P_{1D}
$\ldots$	$\ldots$	$\ldots$	P_{ij}	$\ldots$
A_D	P_{D0}	P_{D1}	$\ldots$	P_{DD}

The first eigenmatrix P

Orthogonal projections:

$$\mathsf{E}_j : \mathbb{R}^V \mapsto \mathbf{W}_j$$

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Two bases of the Bose-Mesner algebra:

$$\langle \mathsf{A}_0, \mathsf{A}_1, \ldots, \mathsf{A}_D \rangle = \langle \mathsf{E}_0, \mathsf{E}_1, \ldots, \mathsf{E}_D \rangle$$

$$\mathsf{E}_j = \frac{1}{|V|} \sum_{i=0}^{D} Q_{ji} \mathsf{A}_i$$

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$G \leq \mathrm{Aut}(\mathcal{S})$: permutations on V that preserve the relations

$g \in G \to$ a permutation matrix $\mathsf{X}_g \in \mathbb{R}^{V \times V}$:

$$\mathsf{X}_g^T = \mathsf{X}_g^{-1} \text{ and } \mathsf{X}_g^n = \mathsf{I} \text{ with } n = |g|, \text{the order of } g,$$

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Every eigenspace $\mathbf{W}_j$ is G-invariant:

$$\mathsf{A}_i(\mathsf{X}_g \overline{w}) = \mathsf{X}_g(\mathsf{A}_i \overline{w}) = P_{ij} \cdot \mathsf{X}_g \overline{w} \text{ for } \overline{w} \in \mathbf{W}_j$$

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Higman's observation

E_j is a projection matrix $\Rightarrow \mathsf{E}_j^2 = \mathsf{E}_j$, and so the eigenvalues of E_j are only 1's and 0's. Now:

$$\begin{aligned}(\mathsf{X}_g\mathsf{E}_j)^{|g|} &= (\mathsf{X}_g)^{|g|}(\mathsf{E}_j)^{|g|} && (\text{by } \mathsf{X}_g\mathsf{E}_j = \mathsf{E}_j\mathsf{X}_g)\\ &= (\mathsf{E}_j)^{|g|} && (\text{by } \mathsf{X}_g^{|g|} = \mathsf{I})\\ &= \mathsf{E}_j\\ &\Rightarrow\end{aligned}$$

non-zero eigenvalues of $\mathsf{X}_g\mathsf{E}_j$ are $\underbrace{\text{roots of unity of order } |g|}$.

$$\underbrace{\text{the sum of eigenvalues}} \in \text{algebraic integers}$$

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$$\mathrm{Trace}(\mathsf{X}_g\mathsf{E}_j) = \frac{1}{|V|}\sum_{i=0}^{D} Q_{ji}\mathrm{Trace}(\mathsf{X}_g\mathsf{A}_i)$$

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is an algebraic integer, and

$$\mathrm{Trace}(\mathsf{X}_g\mathsf{A}_i) = \#\{v \in V \mid (v, v^g) \in R_i\}$$

In particular, if all the eigenvalues P_{ij} are integers:

- all Q_{ji} are rational by $\mathsf{PQ} = |V|\mathsf{I}$,
- $\mathrm{Trace}(\mathsf{X}_g\mathsf{A}_i)$ is an integer by definition,

and thus

$$\mathrm{Trace}(\mathsf{X}_g\mathsf{E}_j) = \frac{1}{|V|}\sum_{i=0}^{D} Q_{ji}\mathrm{Trace}(\mathsf{X}_g\mathsf{A}_i)$$

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Moore graphs

Let Γ be a (undirected) graph:

- regular of valency k,
- of diameter D,
- of (odd) girth γ,
- on N vertices,

then (Hoffman&Singleton, 1960)

$$N \leq 1 + k + k(k-1) + \ldots + k(k-1)^{D-1},$$
$$N \geq 1 + k + k(k-1) + \ldots + k(k-1)^{\frac{\gamma-3}{2}}.$$

If any of these bounds is attained (Damerell, Bannai&Ito):

Diameter	Valency	Moore graph	Transitivity
1	k	K_{k+1}	$\surd$
D	2	C_{2D+1}	$\surd$
2	3	Petersen	$\surd$
2	7	Hoffman-Singleton	$\surd$
2	57	?	$\times$

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A Moore graph of valency 57

Theorem (G. Higman, unpublished)

A Moore graph $\mathcal{M}$ of valency 57 is not vertex-transitive.

Suppose not: the number of vertices is 3250, so $\mathcal{M}$ admits an involution g, and let $\mathrm{Fix}(g)$ be the set of its fixed points.

Step 1. $\mathrm{Fix}(g)$ induces either a star or a Moore subgraph.

Step 2. If $x \sim x^g$ for some vertex x then $|\mathrm{Fix}(g)| = 56$.

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Then $|H|$ is even and so $g \in H \subseteq Alt_{3250}$. But g has 56 fixed points and $\frac{3250-56}{2} = 1597$ transpositions.

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A Moore graph of diameter 2 is strongly regular.

$$\mathsf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 57 & -8 & 7 \\ 3192 & 7 & -8 \end{pmatrix}, \mathsf{Q} = \begin{pmatrix} 1 & 1 & 1 \\ \boxed{1520 \quad -\frac{640}{3} \quad \frac{10}{3}} \\ 1729 & \frac{637}{3} & -\frac{13}{3} \end{pmatrix}$$

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The Higman method

"…The method has not been widely applied, since knowledge of the numbers $\alpha_i(g)$ is not easy to come by."

P. Cameron, "Permutation groups" (1999)

Objectives:

- to prove non-existence of DRGs Γ with given intersection numbers and prescribed symmetries,
- to construct / find all DRGs with prescribed symmetries.

Recipe:

- determine $g \in \mathrm{Aut}(\Gamma)$ of prime order with $\mathrm{Fix}(g) = \emptyset$,
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- determine possible automorphisms of order pq, p^2, etc.
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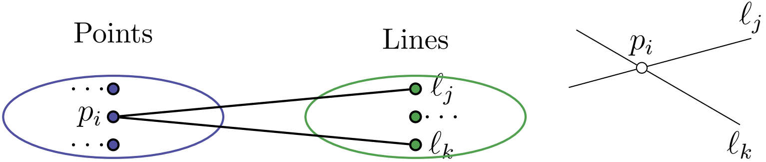
Generalized polygons

The incidence graph of a point-line incidence structure:



Generalized polygons (Tits, 1959):

- the girth is twice the diameter n (a generalized n-gon);
- generalize Moore graphs (the case of even girth);
- of order (s, t) if $\forall$ line has $s + 1$ points and $\forall$ point is on $t + 1$ lines $\Rightarrow n \in \{2, 3, 4, 6, 8\}$ if $s \geq 2, t \geq 2$;
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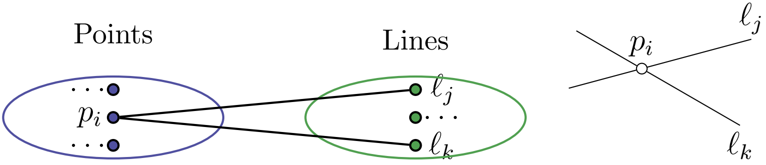

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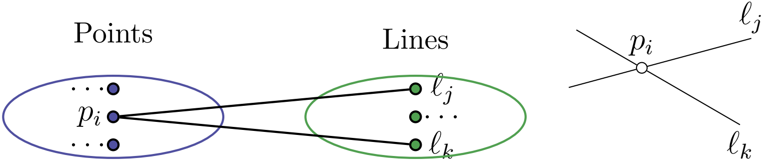

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Theorem (C.T. Benson, 1970)

Let g be an automorphism of a $GQ(s,t)$ with α_0 fixed points and α_1 points x such that x is collinear to x^g. Then

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"… a method of Feit and Higman is extended to provide restrictions on s and t when certain natural automorphisms are present."
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This result has been generalized by Temmermans, Thas, Van Maldeghem (2009) in "On collineations and dualities of finite generalized polygons".

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Various applications of Higman's observation

Theorem (Makhnev, Paduchikh, 2001, 2009)

Let Γ be a Moore graph of valency 57, and $G = \mathrm{Aut}(\Gamma)$. Then $|G| \leq 550$ if $|G|$ is even.

Theorem (Makhnev, Belousov, 2009)

Let Γ be a DRG with intersection array $\{84, 81, 81; 1, 1, 28\}$, and let $\mathrm{Aut}(\Gamma)$ act transitively on the vertex set of Γ. Then $\Gamma \cong GH(3, 27)$ with $\mathrm{Aut}(\Gamma) \cong {}^3D_4(3)$.

Theorem (Makhnev, Belousov, 2008)

Let Γ be a DRG with intersection array $\{10, 8, 8, 8; 1, 1, 1, 5\}$, and let $\mathrm{Aut}(\Gamma)$ act transitively on the vertex set of Γ. Then $\Gamma \cong GO(2, 4)$ with $\mathrm{Aut}(\Gamma) \cong {}^2F_4(2)'$.

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Higman's observation via the character theory

$g \mapsto \mathsf{X}_g$ is a linear representation ρ of G in $\mathbb{R}^V$ with the (permutation) character $\pi(g) = \mathrm{Trace}(\mathsf{X}_g)$.

As every eigenspace $\mathbf{W}_j$ is G-invariant, the restriction

$$\rho|_{\mathbf{W}_j} : G \to GL(\mathbf{W}_j)$$

is a linear representation of G in $\mathbf{W}_j$
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Rational representations

Mačaj and Širáň (2010) further developed this observation:

- if the eigenvalue corresponding to $\mathbf{W}_j$ is integer, then $\mathbf{W}_j$ has a basis over $\mathbb{Q}$;
- A linear representation in $\mathbf{W}_j$ is thus rational;
- $\mathbb{Q}$ has characteristic 0, and by Maschke's Theorem $\rho|_{\mathbf{W}_j}$ is decomposed into rational representations that are irreducible over $\mathbb{Q}$;
- elements x, y of a group H are in the same $\mathbb{Q}$**-class** of $H \Leftrightarrow \langle x \rangle, \langle y \rangle$ are conjugate subgroups of H;
- the number of irreducible $\mathbb{Q}$-representations of $H =$ the number of $\mathbb{Q}$-classes of H.
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For a finite group H, let:

- $x_1, x_2, \ldots, x_u$ be representatives of the $\mathbb{Q}$-classes of H,
- $R_1, R_2, \ldots, R_u$ be the irreducible $\mathbb{Q}$-representations of H with characters $\sigma_1, \sigma_2, \ldots, \sigma_u$.

Then, for any rational representation of H with character χ, the system of linear equation with the matrix

$$\left(\begin{array}{ccc|c} \sigma_1(x_1) & \ldots & \sigma_u(x_1) & \chi(x_1) \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_1(x_u) & \ldots & \sigma_u(x_u) & \chi(x_u) \end{array}\right)$$

has a solution in non-negative integers $c_1, c_2, \ldots, c_u$:

$$\chi = c_1\sigma_1 + \ldots + c_u\sigma_u.$$

For example, if $|g| = p$ and $m_j = \mathrm{Rank}(\mathsf{E}_j) = \chi_j(1)$:

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Update on the Moore valency 57 graph problem

$$\chi_j(g) = \text{Trace}(\mathsf{X}_g \mathsf{E}_j) = \frac{1}{|V|} \sum_{i=0}^{D} Q_{ji} \alpha_i(g)$$

so

$$(\chi_0(g), \chi_1(g), \ldots, \chi_D(g))^T = \frac{1}{|V|} \mathsf{Q} \cdot (\alpha_0(g), \alpha_1(g), \ldots, \alpha_D(g))^T$$

Corollary (Mačaj and Širáň, 2010)

If all eigenvalues of an association scheme are integral, then the functions $\alpha_i(g)$ are constant on rational classes. In particular, $\alpha_i(g) = \alpha_i(g^2) = \ldots = \alpha_i(g^{|g|-1})$ if $|g|$ is a prime.

Theorem (Mačaj and Širáň, 2010)

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In 1998, Godsil, Liebler, Praeger classified antipodal distance-transitive covers of complete graphs. Their key observation was that $\mathrm{Aut}(\Gamma)$ of such a graph Γ induces a 2-transitive action on the set of fibres.

Tsiovkina noticed that it is enough to assume that $\mathrm{Aut}(\Gamma)$ acts transitively on arcs of Γ.

In a series of papers (2013-2017), Makhnev, Paduchikh and Tsiovkina classified arc-transitive distance-regular covers of complete graphs.

Besides distance-transitive graphs, they found several new examples including three infinite series related to $Sz(q)$, $SU_3(q)$ and ${}^2G_2(q)$.

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More recent results: partial difference sets

Theorem (De Winter, Kamischke, Wang, 2014)

Let Γ be an SRG on v vertices whose adjacency matrix has integer eigenvalues k, ν_2 and ν_3. Let g be an automorphism of order n of Γ, and let $\mu()$ be the Möbius function. Then for every integer r and all positive divisors d of n, there are non-negative integers a_d and b_d such that

$$k-r+\sum_{d|n} a_d\mu(d)(\nu_2-r)+\sum_{d|n} b_d\mu(d)(\nu_3-r) = -r\alpha_0(g)+\alpha_1(g)$$

Moreover, $a_1+b_1=c-1$, where c is the number of cycles in the disjoint cycle decomposition of g, and $a_d+b_d=\sum_{d|\ell} c_\ell$, $d\neq 1$, where c_ℓ is the number of cycles of length ℓ of g.

Using this, they ruled out several open cases of partial difference sets in abelian groups (from the list of feasible parameters by S.L. Ma).

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One more Moore graphs problem

The Moore bounds

$$N \leq 1 + k + k(k-1) + \ldots + k(k-1)^{D-1},$$
$$N \geq 1 + k + k(k-1) + \ldots + k(k-1)^{\frac{\gamma-3}{2}}.$$

can be generalized to the cases of directed (arcs only) and mixed (with both edges and arcs) graphs.

- the only Moore digraph of diameter > 1 is $\overrightarrow{C_3}$;
  Plesnik, Znám (1974)
- no mixed Moore graph can exist for diameters > 2
  Nguyen, Miller, Gimbert (2007)
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Directed (mixed?) strongly regular graphs

DSRG(v, k, t, λ, μ):

- the total vertex degree is $k = t + z$:
  - every vertex is incident to t edges;
  - every vertex is incident to z in-arcs and z out-arcs;
- for every arc/edge $a \rightharpoonup b$ there exist λ vertices c such that $a \rightharpoonup c \rightharpoonup b$;
- for every non-arc $a \not\rightharpoonup b$ there exist μ vertices c such that $a \rightharpoonup c \rightharpoonup b$;

The adjacency matrix A defined by $(\mathsf{A})_{a,b} = 1$ if $a \rightharpoonup b$:

$$\mathsf{A}^2 = t\mathsf{I} + \lambda\mathsf{A} + \mu(\mathsf{J} - \mathsf{I} - \mathsf{A}),$$
$$\mathsf{AJ} = \mathsf{JA} = k\mathsf{J}.$$

A is diagonalizable with 3 eigenspaces:

$$\mathsf{A} \in \langle \mathsf{E}_0, \mathsf{E}_1, \mathsf{E}_2 \rangle \text{ and } \mathsf{E}_j \in \langle \mathsf{A}, \mathsf{I}, \mathsf{J} \rangle$$

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Jørgensen (2003); Godsil, Hobart, Martin (2007)

Directed (mixed?) strongly regular graphs

DSRG(v, k, t, λ, μ):

- the total vertex degree is $k = t + z$:
  - every vertex is incident to t edges;
  - every vertex is incident to z in-arcs and z out-arcs;
- for every arc/edge $a \rightharpoonup b$ there exist λ vertices c such that $a \rightharpoonup c \rightharpoonup b$;
- for every non-arc $a \not\rightharpoonup b$ there exist μ vertices c such that $a \rightharpoonup c \rightharpoonup b$;

The adjacency matrix A defined by $(\mathsf{A})_{a,b} = 1$ if $a \rightharpoonup b$:

$$\mathsf{A}^2 = t\mathsf{I} + \lambda\mathsf{A} + \mu(\mathsf{J} - \mathsf{I} - \mathsf{A}),$$
$$\mathsf{AJ} = \mathsf{JA} = k\mathsf{J}.$$

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Mixed Moore graphs

A mixed Moore graph ($\not\cong C_5$ or $\overrightarrow{C_3}$) is a DSRG$(v, k, t, 0, 1)$, $t = \frac{c^2+3}{4}$ for an odd integer $c > 0$ with $c|(4z-3)(4z+5)$.

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(For every pair of vertices a, b, there is a unique path $a \rightharpoonup \ldots \rightharpoonup b$ of length at most 2.)

The known Mixed Moore graphs:

- $t = 1$: the Kautz digraphs;

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- $t > 1$: only three graphs are known:
  - the Bosák graph $(18, 4, 3, 0, 1)$;
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- All three known graphs with $t > 1$ are Cayley graphs;
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Re-define:

$$\alpha_0(g) := \mathrm{Trace}(\mathsf{X}_g \mathsf{I}) = \#\{x \in V \mid x = x^g\},$$
$$\alpha_1(g) := \mathrm{Trace}(\mathsf{X}_g \mathsf{A}) = \#\{x \in V \mid x \rightharpoonup x^g\},$$

so that $\mathrm{Trace}(\mathsf{X}_g \mathsf{E}_j)$ is a linear combination of $\alpha_0(g), \alpha_1(g)$.

The eigenvalues of DSRGs are always integral, so by applying $\mathbb{Q}$-representation theory we obtain that the functions $\alpha_i(g)$ are constant on rational classes.

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Mixed Moore-Cayley graphs

Example:
suppose that a DSRG$(88, 9, 3, 0, 1)$ is a Cayley graph.
Then $\mathrm{Aut}(\Gamma)$ has an element g with $|g| = 11$ and $\alpha_0(g) = 0$.
By Higman's observation, $\alpha_1(g) \in \{11, 44, 77\}$.

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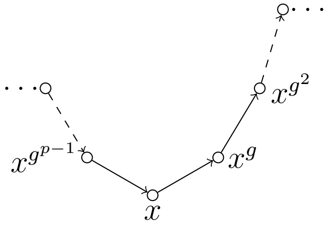

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Suppose $\alpha_1(g) = 11$:

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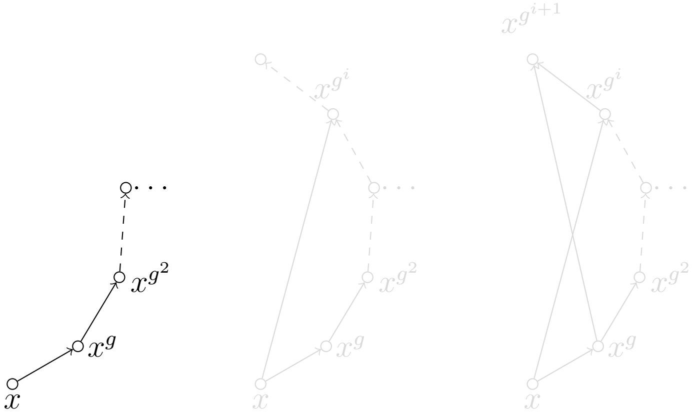

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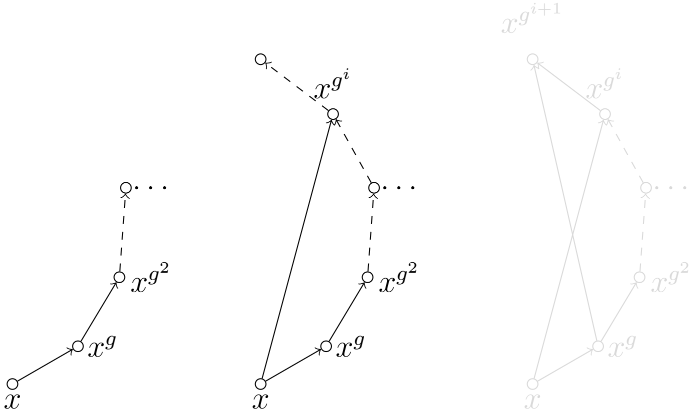

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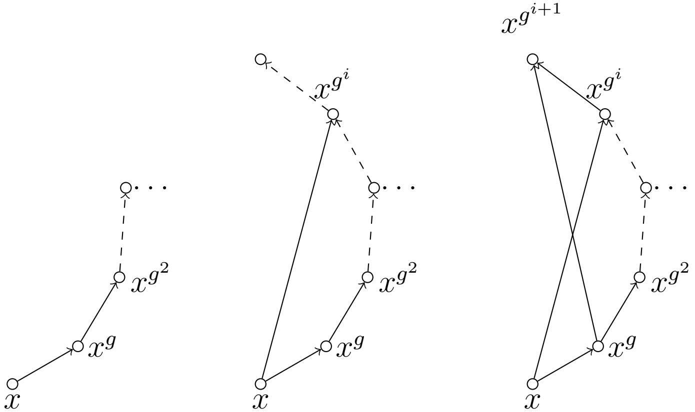

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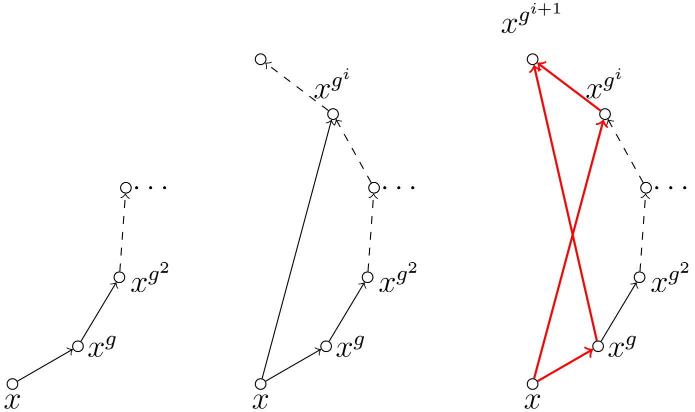
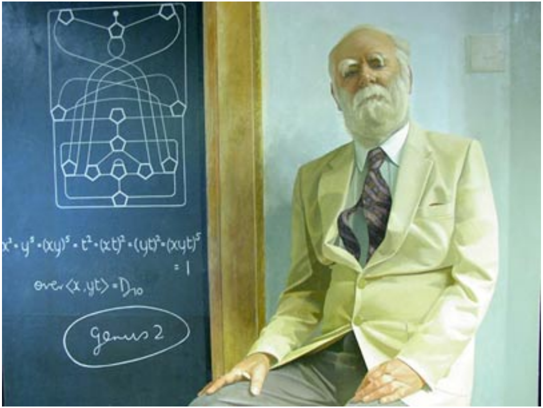


Mixed Moore-Cayley graphs

This rules out mixed Moore-Cayley graphs of orders 88, 204, 238, 368, 460, $\ldots$.

Graham Higman

Thank you!



(by Norman Blamey, 1984)