$PI$-eigenfunctions of the Star graphs

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Outline

1. Preliminary
   • Cayley graph \( \text{Cay}(G, S) \) and the neighbourhood of a vertex
   • Group algebra \( \mathbb{F}[G] \) and the action by multiplication
   • A bridge between \( \text{Cay}(G, S) \) and \( \mathbb{F}[G] \)
   • The bridge between the Star graph \( S_n \) and the Jucys-Murphy element \( J_n \)
   • A result by Jucys on eigenvectors of \( J_n \)
   • Permutation module \( M^\lambda \) and Specht module \( S^\lambda \)
   • An embedding of the permutation module \( M^\lambda \) into \( \mathbb{C}[\text{Sym}_n] \)
   • Standard basis of the Specht module and eigenvectors of \( J_n \) given by a polytabloid

2. Our results
   • A family of eigenfunctions of the Star graph \( S_n \) called \( PI \)-eigenfunctions
   • A connection between eigenfunctions given by a polytabloid and \( PI \)-eigenfunctions
Let $G$ be a group. For a non-empty inverse-closed identity-free subset $S$ in $G$, define the Cayley graph $Cay(G, S)$ with the vertex set $G$ and two vertices $x, y$ being adjacent whenever there is an element $s \in S$ such that $y = sx$ holds.

Given a vertex $x$ in $Cay(G, S)$, the equality

$$N(x) = Sx$$

holds, where $N(x)$ is the neighbourhood of $x$. 
Take a field $\mathbb{F}$, a group $G$ and the group algebra $\mathbb{F}[G]$.

For a subset $S$ in $G$, consider the element $\overline{S} \in \mathbb{F}[G]$ given by

$$\overline{S} = \sum_{s \in S} s.$$ 

Left multiplication of elements from $\mathbb{F}[G]$ by $\overline{S}$ is a linear transformation of $\mathbb{F}[G]$.

The transformation matrix of this linear transformation coincides with the adjacency matrix of $Cay(G, S)$, which gives a bridge between spectral properties of Cayley graphs and the representation theory.
Let $\Gamma = (V, E)$ be a regular graph.

A function $f : V \to \mathbb{R}$ is called an **eigenfunction** of the graph $\Gamma$ corresponding to an eigenvalue $\theta$ if $f \not\equiv 0$ and for any vertex $x$ the local condition

$$\theta \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds, where $N(x)$ is the set of neighbours of the vertex $x$. 
We put

- $G := Sym_n$
- $F := \mathbb{C}$
- $S := \{(i \ n) \mid i = 1, \ldots, n - 1\}$
- The graph $S_n := Cay(G, S)$ is called the Star graph
- The element $J_n := (1 \ n) + \ldots + (n - 1 \ n)$ from $\mathbb{C}[Sym_n]$ is called the Jucys-Murphy element
Partitions and Ferrers diagrams

Take the partition $\lambda = (4, 2, 1)$ of the number $n = 7$. Then the corresponding Ferrers diagram is as follows:

![Ferrers Diagram](image)

A tableau $t$ is **standard** if the rows and columns of $t$ are increasing sequences.
Standard Young tableaux of shape $\lambda = (4, 2, 1)$

$c(n) = 3$

\[
\begin{array}{cccc}
1 & 2 & 6 & 7 \\
3 & 5 & 4 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 5 & 7 \\
3 & 6 & 4 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 6 & 7 \\
3 & 4 & 5 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & 7 \\
3 & 6 & 5 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 5 & 7 \\
3 & 4 & 6 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & 7 \\
3 & 5 & 6 \\
\end{array}
\]

$c(n) = 0$

\[
\begin{array}{cccc}
1 & 2 & 5 & 6 \\
3 & 7 & 4 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & 6 \\
3 & 7 & 5 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & 5 \\
3 & 7 & 6 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 5 & 6 \\
2 & 7 & 4 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 4 & 6 \\
2 & 7 & 5 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 4 & 5 \\
2 & 7 & 6 \\
\end{array}
\]

$c(n) = -2$

\[
\begin{array}{cccc}
1 & 2 & 5 & 6 \\
3 & 4 & 7 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & 6 \\
3 & 5 & 7 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & 5 \\
3 & 6 & 7 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 5 & 6 \\
2 & 4 & 7 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 4 & 6 \\
2 & 5 & 7 \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 4 & 5 \\
2 & 6 & 7 \\
\end{array}
\]
The regular representation of $\text{Sym}_n$ is decomposed into irreducible submodules as follows

$$\mathbb{C}[\text{Sym}_n] = \bigoplus_{\lambda \in \mathcal{P}(n)} m_\lambda V_\lambda,$$

where $\mathcal{P}(n)$ the set of partitions of $n$ and $m_\lambda = \dim V_\lambda$. 
Theorem (Jucys, 1974)

Let \( \lambda \in \mathcal{P}(n) \). Then there exists a basis \( \{v_t\} \) of the irreducible module \( V_{\lambda} \), indexed by standard Young tableaux \( t \) of shape \( \lambda \) such that for all \( i \in \{2, \ldots, n\} \), the equality

\[
J_i v_t = c_t(i)v_t
\]

holds.

If \( i = n \), the theorem says that there exists a basis of an irreducible module \( V_{\lambda} \) consisting of eigenvectors of \( J_n \). Moreover, the number of eigenvectors corresponding to the same eigenvalue is given by the number of standard Young tableaux of the shape \( \lambda \) with the same value \( c(n) \).
The irreducible module $V_{\lambda}$ has dimension 24.

The basis contains

- 12 eigenvectors of $J_7$ with eigenvalue 3,
- 6 eigenvectors with eigenvalue 0,
- 6 eigenvectors with eigenvalue $-2$. 

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Corollary

For any \( n \geq 4 \), the spectrum of the Star graph \( S_n \) consists of integers \(-(n - 1), \ldots, n - 1\), and the multiplicity of an eigenvalue \( \theta \) is given by the formula

\[
\text{mul}(\theta) = \sum_{\lambda \in \mathcal{P}(n)} m_\lambda \cdot C_\lambda(k),
\]

where \( m_\lambda \) is the number of standard Young tableaux of shape \( \lambda \) and \( C_\lambda(k) \) is the number of standard Young tableaux of shape \( \lambda \) with \( c(n) = k \).
Let $\lambda$ be a shape with $n$ cells.

For a tableau $t$ of shape $\lambda$, the $\lambda$-tabloid $\{t\}$ is the set of all tableaux of shape $\lambda$ that can be obtained from $t$ by permutations of elements in rows.

Let $M^\lambda = \mathbb{C}\{\{t_1\}, \ldots, \{t_k\}\}$ be the permutation module corresponding to $\lambda$, where $\{t_1\}, \ldots, \{t_k\}$ is a complete list of $\lambda$-tabloids.

Further, we consider the action of the group algebra $\mathbb{C}[\text{Sym}_n]$ on $M^\lambda$. 
Let $t$ be a tableau of shape $\lambda$.

Let $C_t$ be the column stabilizer of $C_t$.

Put

$$e_t := \sum_{\pi \in C_t} \text{sgn}(\pi)\{\pi(t)\}.$$

The element $e_t \in M^\lambda$ is called the polytabloid given by the tableau $t$. 

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Given a partition $\lambda$, the corresponding Specht module $S^\lambda$, is the submodule of $M^\lambda$ spanned by all polytabloids $e_t$, where $t$ is of shape $\lambda$.

A polytabloid $e_t$ is standard if the tableau $t$ is standard.

The set of standard polytabloids

$$\{e_t \mid t \text{ is a standard tableau of shape } \lambda\}$$

forms a basis of the Specht module $S^\lambda$. 
An embedding $\phi$ of $M^\lambda$ into $\mathbb{C}[\text{Sym}_n]$

Let $id_\lambda$ be the standard tableau of shape $\lambda$ whose rows consist of the consecutive elements.

Let $T_\lambda$ be the set of all tableaux of shape $\lambda$. For any tableau $t \in T_\lambda$, denote by $\tau_t$ the permutation defined by the equation

$$\tau_t(t) = id_\lambda,$$  \hspace{1cm} (1)

where $\tau_t$ acts on $t$ by replacing the values of the cells of $t$.

Let us define a linear mapping $\phi : M^\lambda \to \mathbb{C}[\text{Sym}_n]$. Since the set of all $\lambda$-tabloids is a basis for $M_\lambda$, it is enough to define images for $\lambda$-tabloids. For any $\lambda$-tabloid $\{t\}$, where $t \in T_\lambda$, we put

$$\phi(\{t\}) = \sum_{t' \in \{t\}} \tau_{t'}.$$

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Lemma (1)

For any polytabloid $e_t$, the equality $\phi(J_n(e_t)) = J_n(\phi(e_t))$ holds.

Lemma (2)

Let $v \in S^\lambda$ be an eigenfunction of the operator $J_n : M^\lambda \to M^\lambda$ with eigenvalue $\theta$. Then $\phi(v)$ is an eigenfunction of the operator $J_n : \mathbb{C}[\text{Sym}_n] \to \mathbb{C}[\text{Sym}_n]$ with eigenvalue $\theta$. 
An eigenvector of $J_n$ given by a polytabloid $e_t$

Let $\lambda \in \mathcal{P}(n)$ be a partition $(\lambda_1, \lambda_2, \ldots, \lambda_s)$, where $s \geq 2$, $\lambda_1 > \lambda_2$ and $\lambda_i \geq \lambda_{i+1}$ for any $i \in \{2, \ldots, s-1\}$. Put $m = \lambda_2 + \ldots + \lambda_s$. In this setting $m$ is the number of cells in all rows of $\lambda$ but the first.

Let $t$ be a standard tableau of shape $\lambda$ with $n$ placed at its upper right cell.

Lemma (3)

The polytabloid $e_t$ is an eigenfunction of $J_n$ with eigenvalue $n - m - 1$. 

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Main result 1: the family of $PI$-eigenfunctions of $S_n$

Let us take a vector

$$P_m = ((j_1, k_1), (j_2, k_2), \ldots, (j_m, k_m))$$

of $2m$ pairwise different elements from the set $\{1, \ldots, n - 1\}$ arranged into $m$ pairs and a vector

$$I_m = (i_1, i_2, \ldots, i_m)$$

of $m$ pairwise different elements from the set $\{1, \ldots, n\}$. Define a function $f_{I_m}^{P_m} : \text{Sym}_n \rightarrow \mathbb{R}$. For a permutation

$$\pi = [\pi_1 \pi_2 \ldots \pi_n] \in \text{Sym}_n,$$

we put

$$f_{I_m}^{P_m}(\pi) = 0,$$

if there exists $t \in \{1, 2, \ldots, m\}$ such that $\pi_{j_t} \neq i_t$ and $\pi_{k_t} \neq i_t$. If for every $t \in \{1, 2, \ldots, m\}$ either $\pi_{j_t} = i_t$ or $\pi_{k_t} = i_t$, then we define a binary vector $X_\pi = (x_1, x_2, \ldots, x_m)$ as follows:

$$x_t = \begin{cases} 1, & \text{if } \pi_{j_t} = i_t; \\ 0, & \text{if } \pi_{k_t} = i_t. \end{cases}$$
Main result 1: the family of $PI$-eigenfunctions of $S_n$

We use the vector $X_\pi$ to complete the definition of the function $f_{I_m}^{P_m}$:

$$f_{I_m}^{P_m}(\pi) = \begin{cases} 
1, & \text{if } X_\pi \text{ contains an even number of 1s;} \\
-1, & \text{if } X_\pi \text{ contains an odd number of 1s;} \\
0, & \text{there exists } t \text{ such that } \pi_{j_t} \neq i_t \text{ and } \pi_{k_t} \neq i_t.
\end{cases}$$

**Theorem (1)**

For $n \geq 3$, the function $f_{I_m}^{P_m}$ is an eigenfunction with eigenvalue $n - m - 1$ of the Star graph $S_n$. 
Main result 2: an expression of an eigenfunction given by a polytabloid in $PI$-eigenfunctions

Let $e_t$ be an eigenfunction from the Lemma 3, and $n > 2m$ holds.

For any $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, k\}$, denote by $R_t(i)$ and $C_t(j)$ the symmetric groups on the elements of $i$th row and $j$th column of the tableau $t$, respectively. Then we have

$$R_t = R_t(1) \times R_t(2) \times \ldots \times R_t(s),$$

$$C_t = C_t(1) \times C_t(2) \times \ldots \times C_t(k),$$

where $R_t$ and $C_t$ are the row-stabilizer and the column-stabilizer of $t$, respectively.
Main result 2: an expression of an eigenfunction given by a polytabloid in $PI$-eigenfunctions

Put

$$CA_t = CA_t(1) \times CA_t(2) \times \ldots \times CA_t(k),$$

where $CA_t(j)$ denotes the subgroup of even permutations in $C_t(j)$.

**Theorem (2)**

The equality

$$f_{\phi(e_t)} = \sum_{\sigma \in R_t(2) \times \ldots \times R_t(s)} \sum_{\pi \in CA_{\sigma(t)}} f^{P_{\pi}}_{(n-m+1,...,n)}$$

holds, where $P_{\pi}$ is a vector of $m$ pairs uniquely determined by $\pi$. 
Questions

1. Can we construct linearly independent embeddings of a Shecht module into $\mathbb{C}[\text{Sym}_n]$?
2. Can we write down explicitly a basis of an eigenspace of the Star graph?
3. Is the family of $PI$-eigenfunctions complete? It is true in the case of the largest non-principal eigenvalue $n - 2$ of $S_n$. Moreover, we can find a basis among $PI$-eigenfunctions.
4. Does there exist an analogue of the $PI$-eigenfunctions for the other half of positive eigenvalues of the Star graph?
Thank you for your attention!