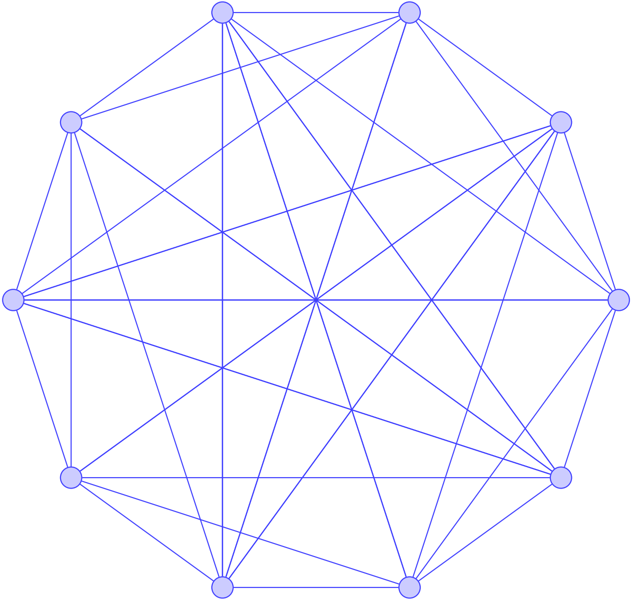
Edge-regular graphs and regular cliques

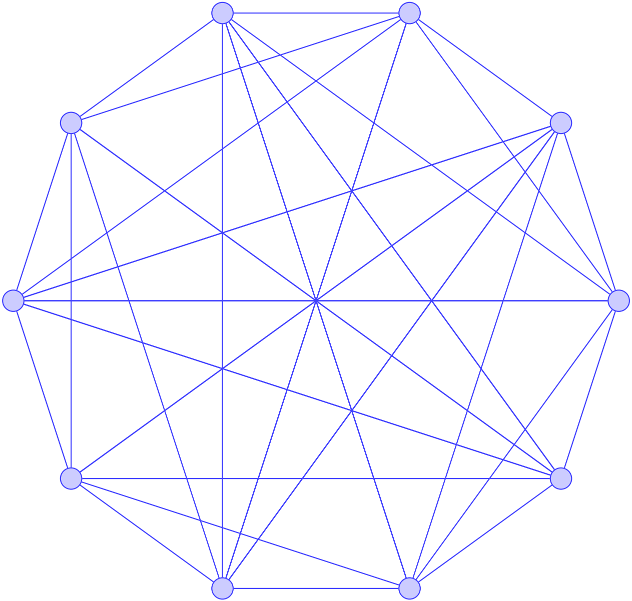
Gary Greaves

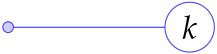
Nanyang Technological University, Singapore

6th July 2018

joint work with J. H. Koolen



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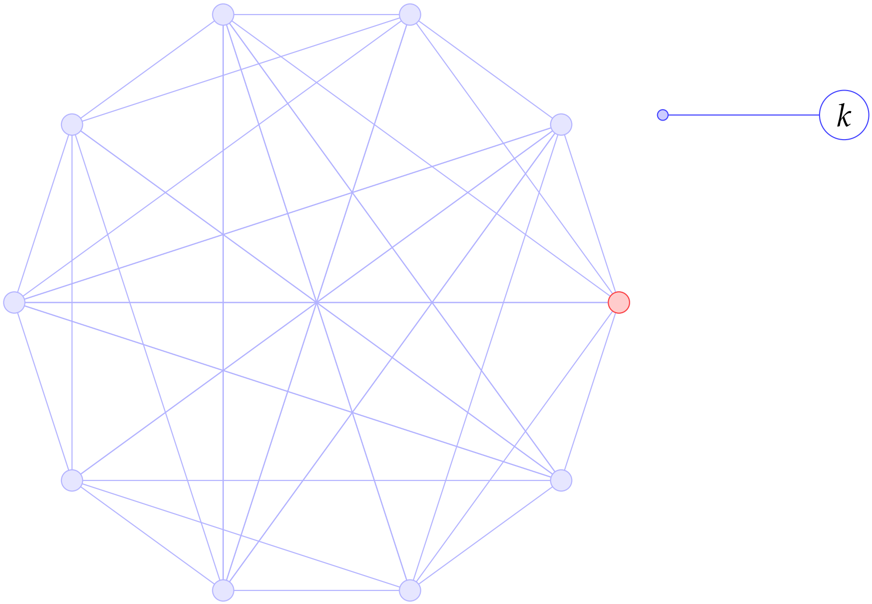
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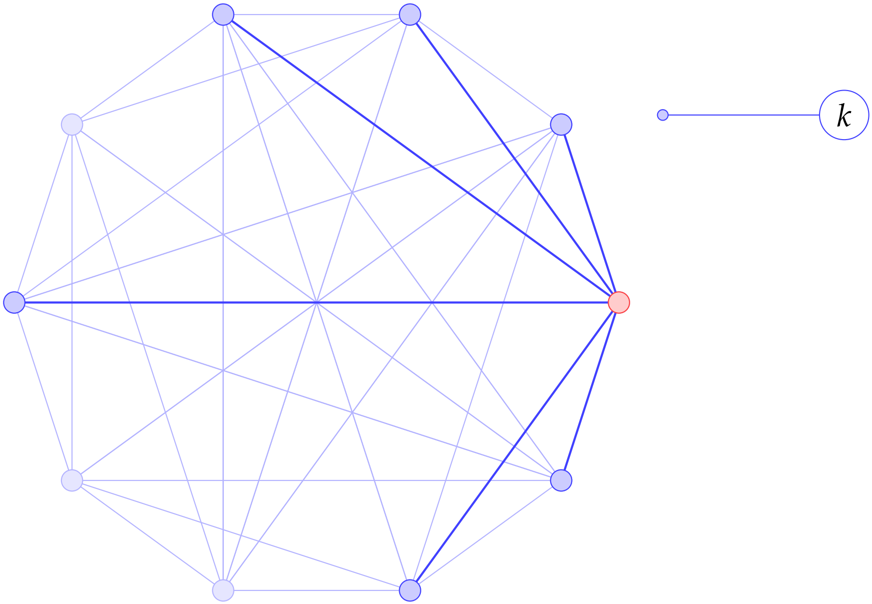

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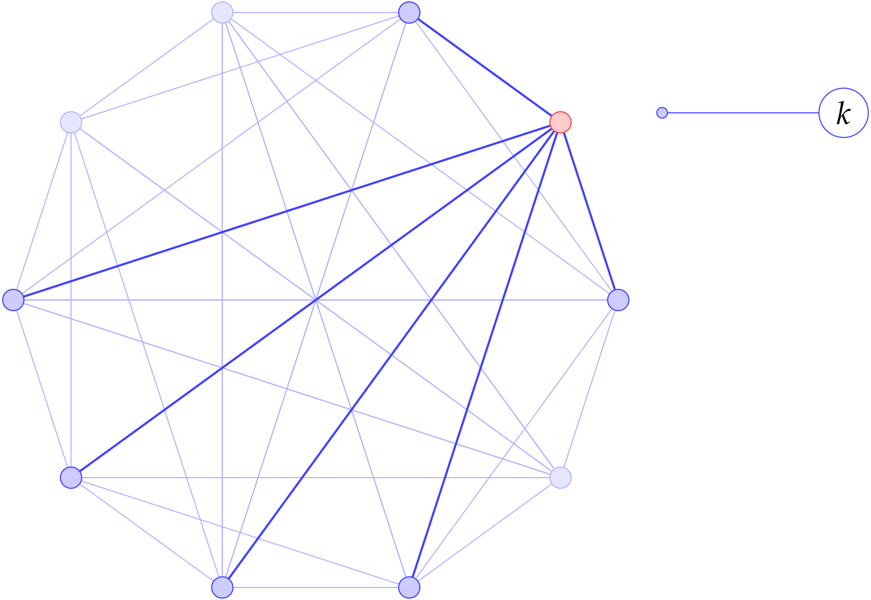
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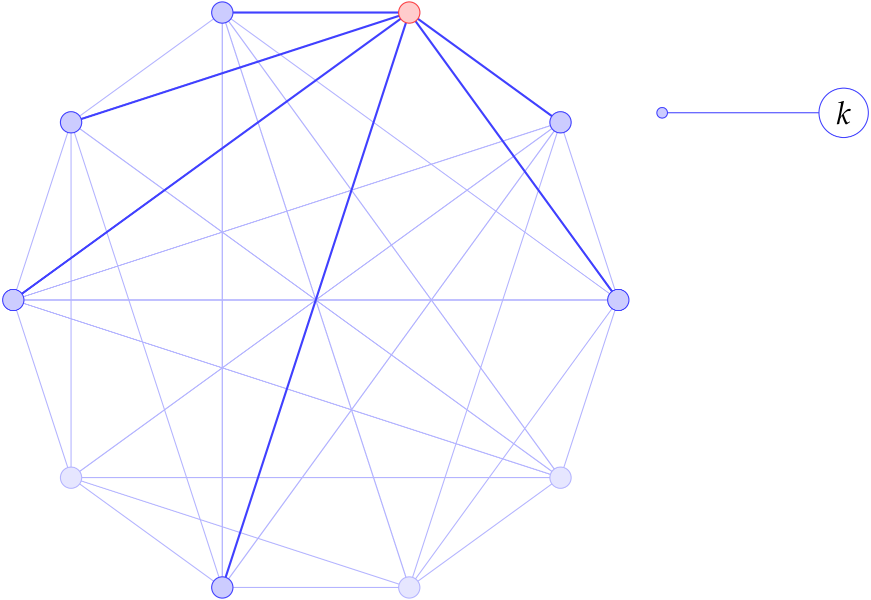
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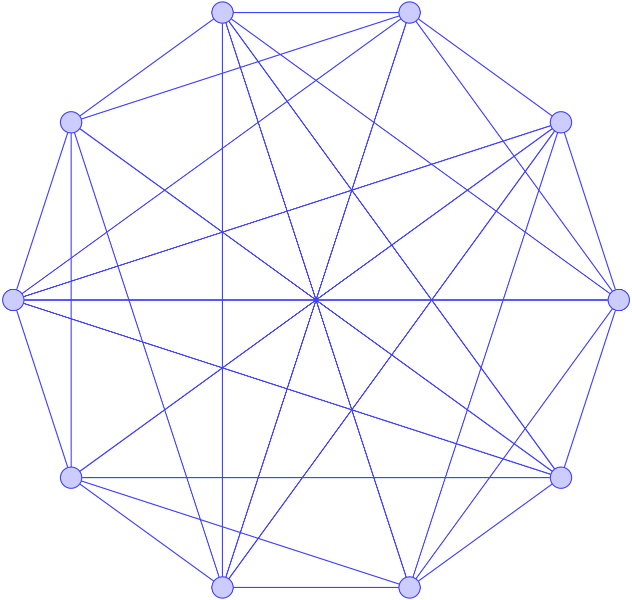

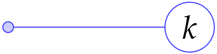

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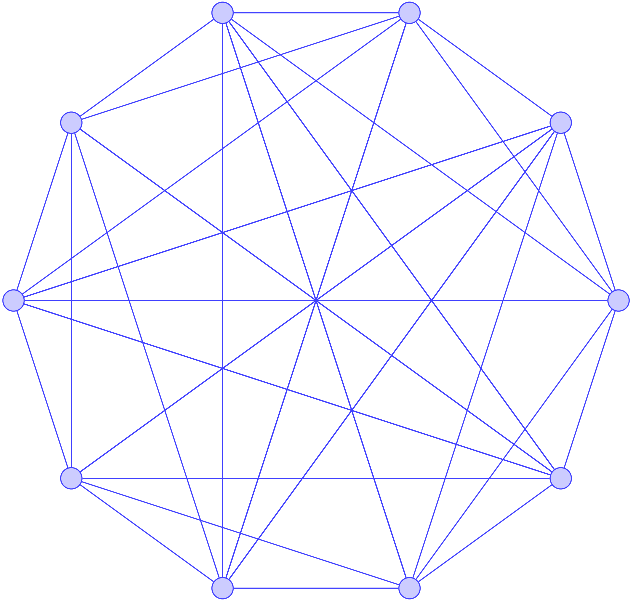

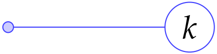

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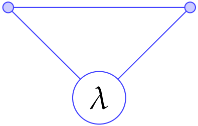

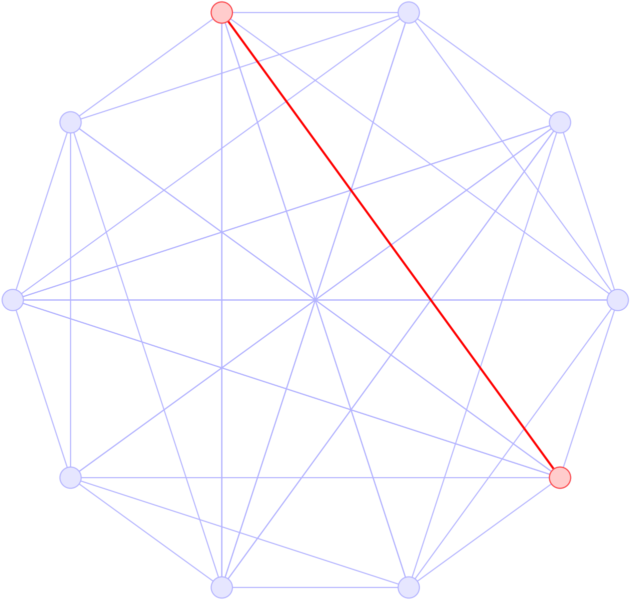

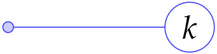

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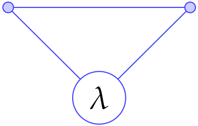

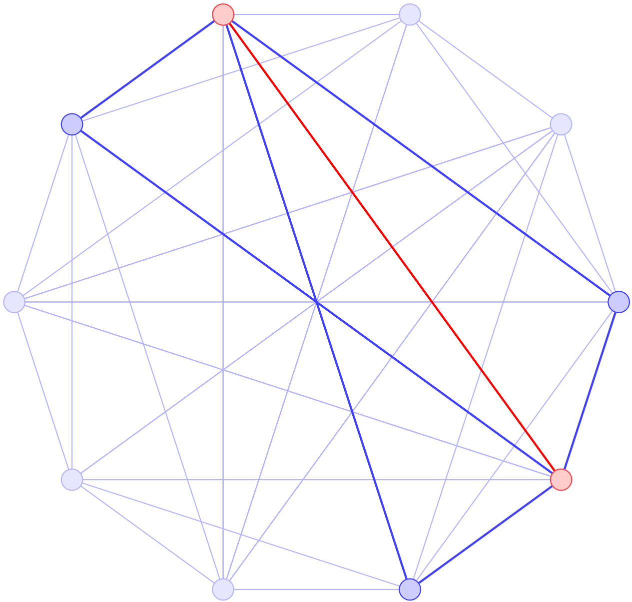

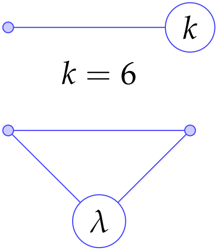
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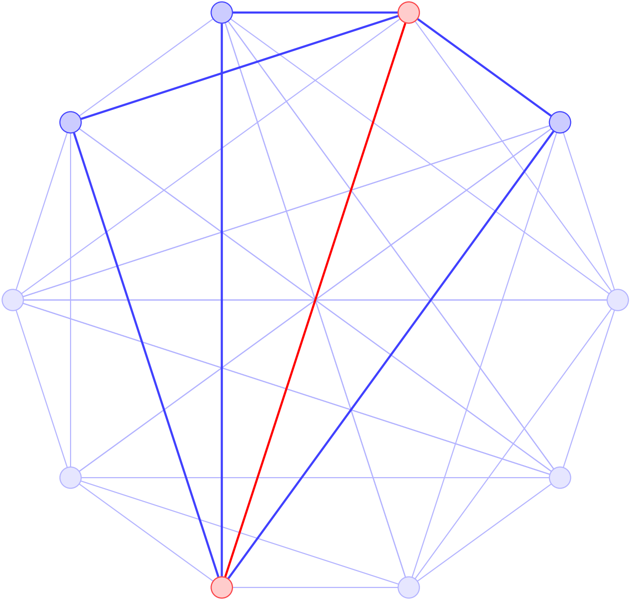

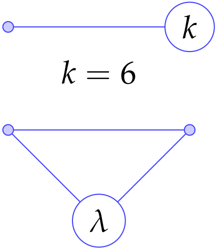
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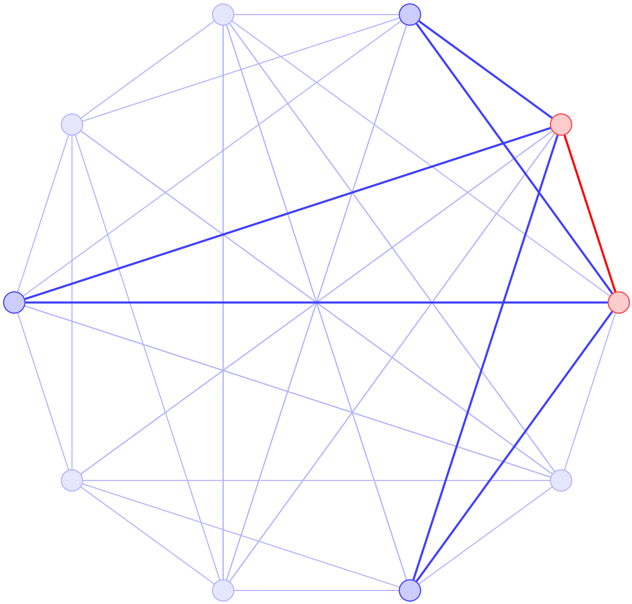

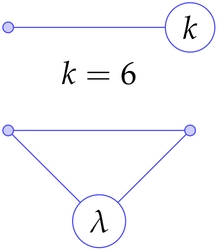
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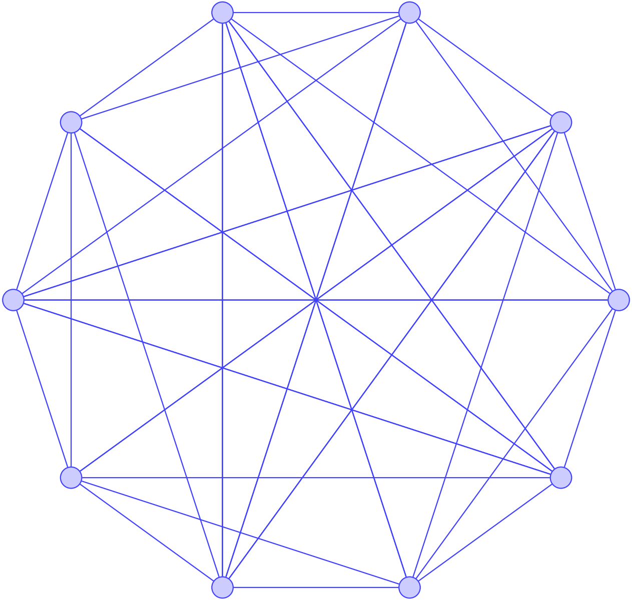

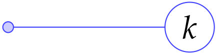

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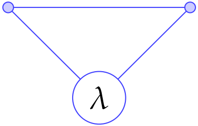
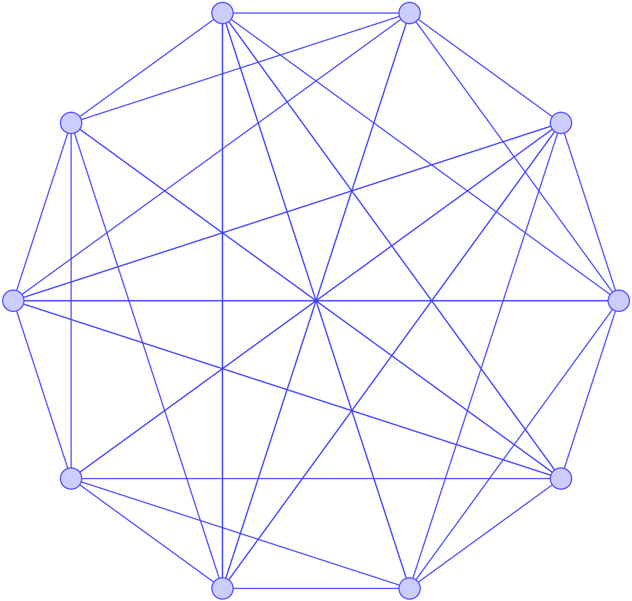
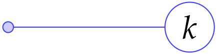


$\lambda = 3$





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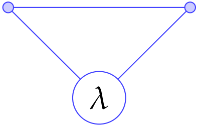

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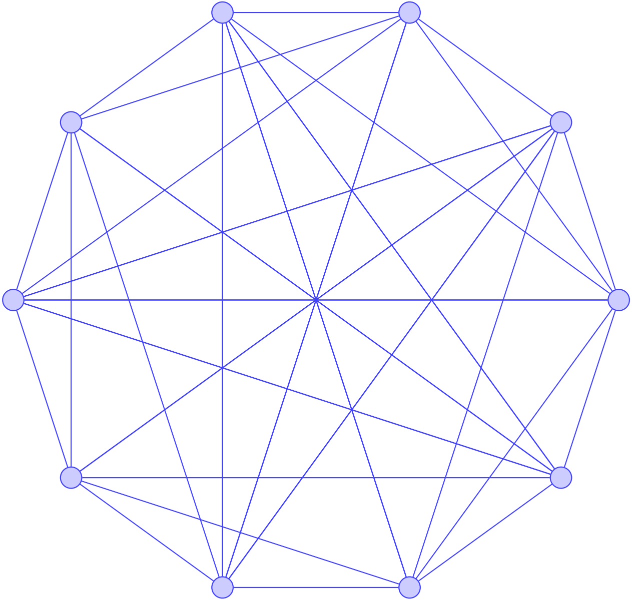
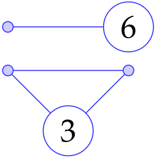
edge-regular

$\mathrm{erg}(10, 6, 3)$





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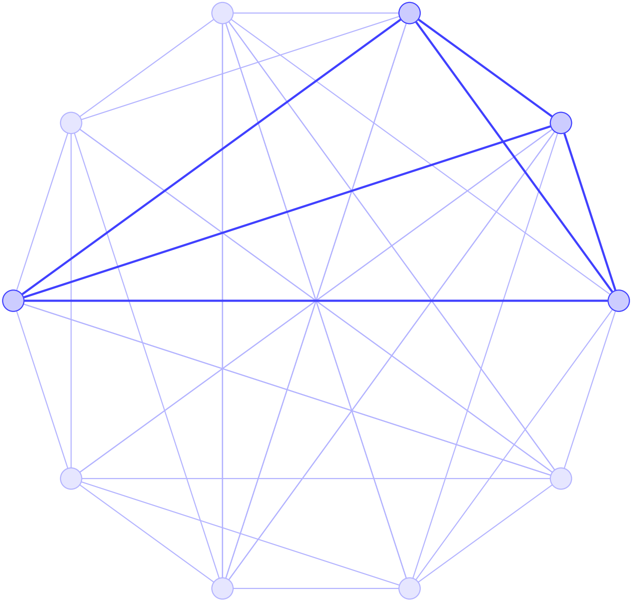

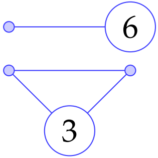

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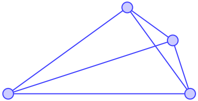
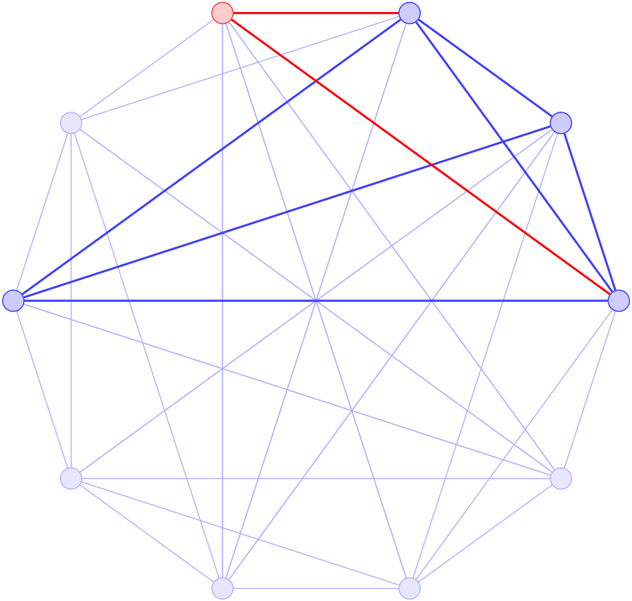
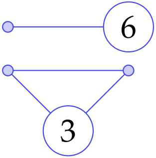


clique

of order 4





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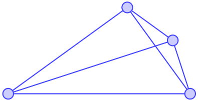

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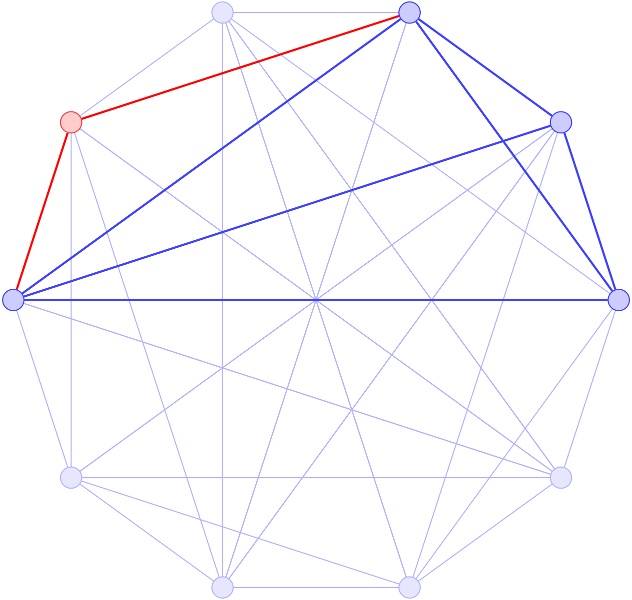

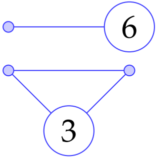

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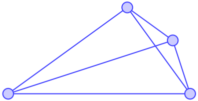

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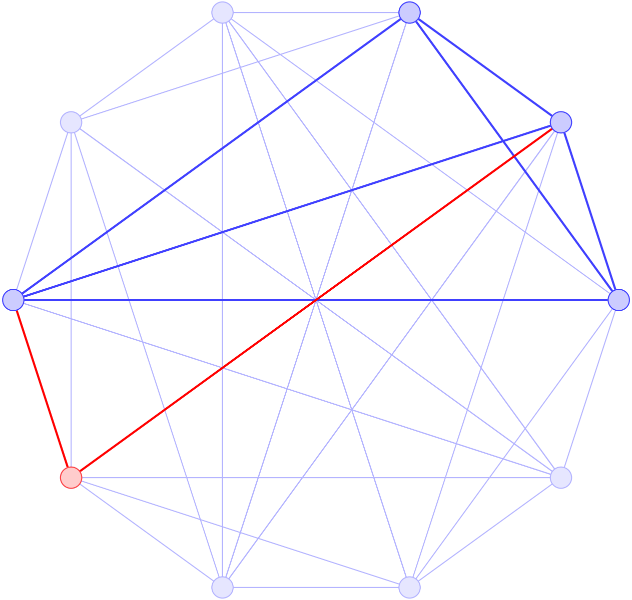

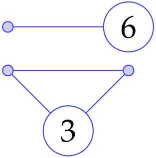

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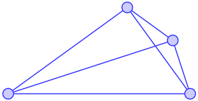

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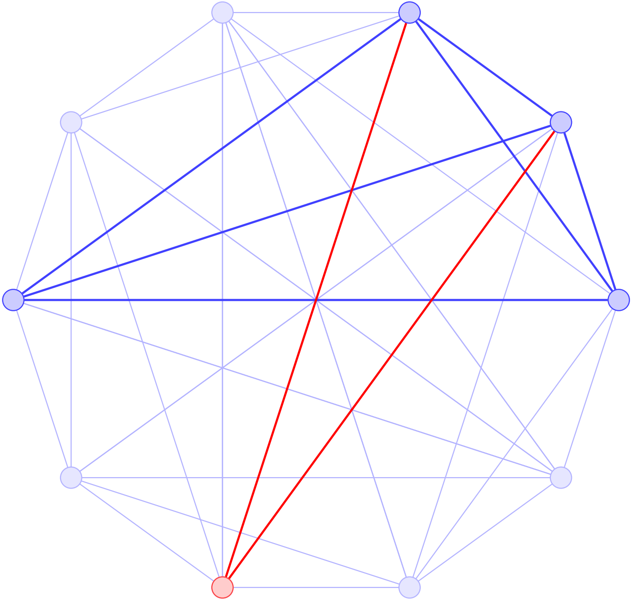

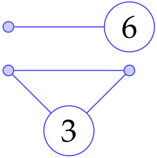

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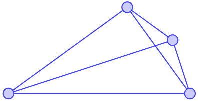

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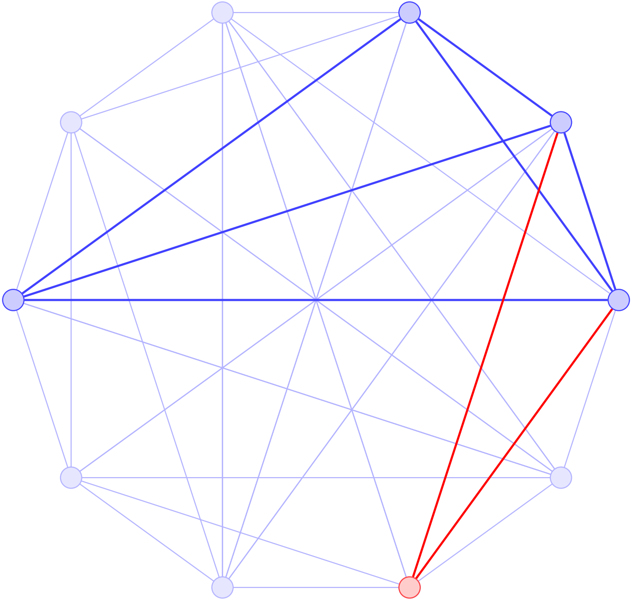

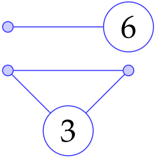

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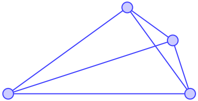

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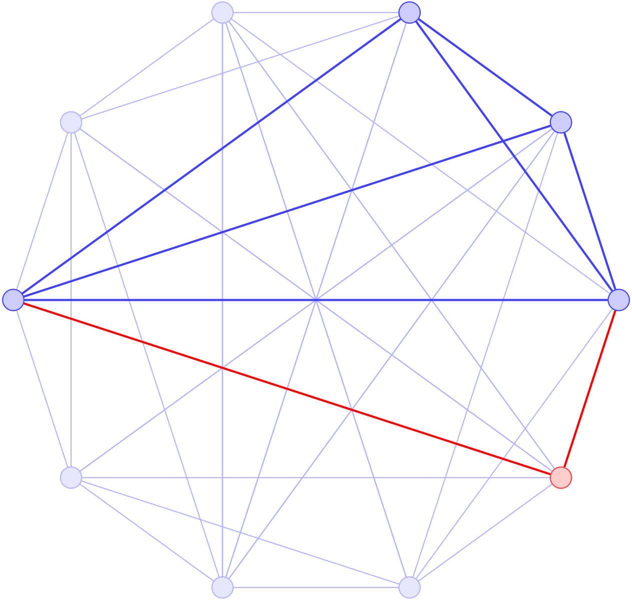

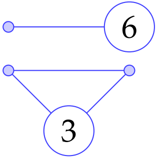

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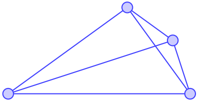

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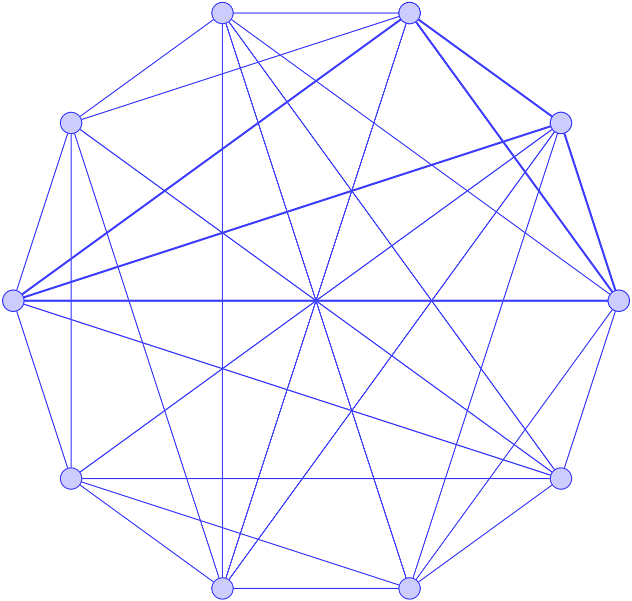

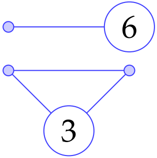

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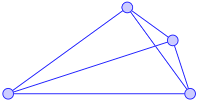

2-**regular** clique

of order 4

Theorem (Neumaier 1981)

Let Γ be edge-regular with a regular clique.
Suppose Γ is vertex-transitive and edge-transitive.
Then Γ is strongly regular.

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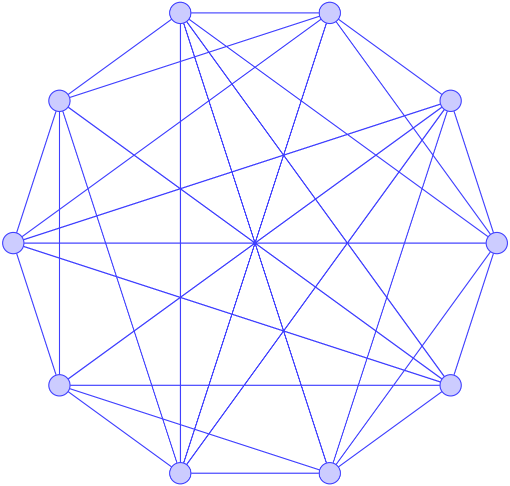

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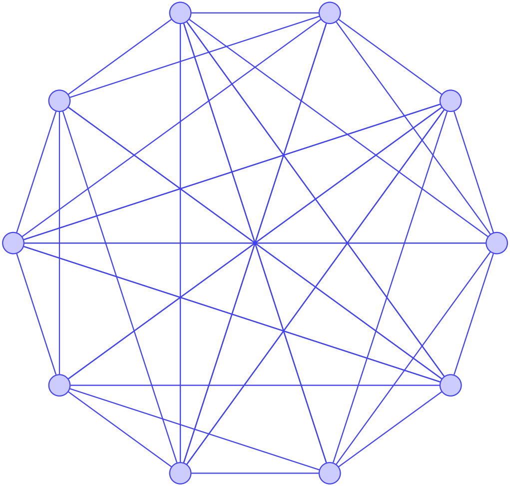

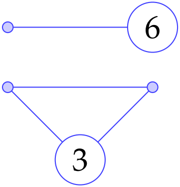

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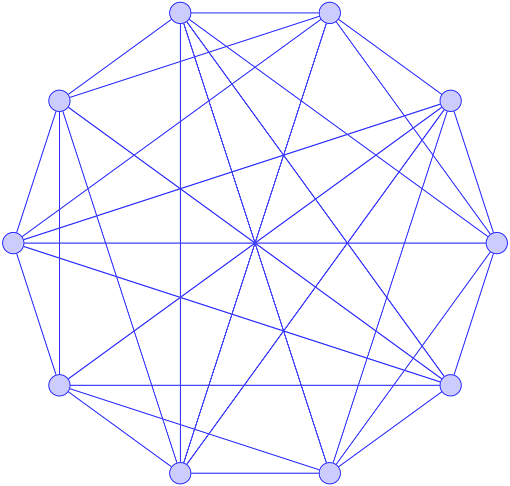

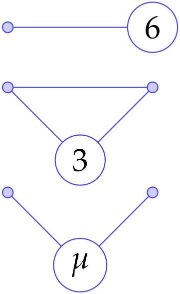

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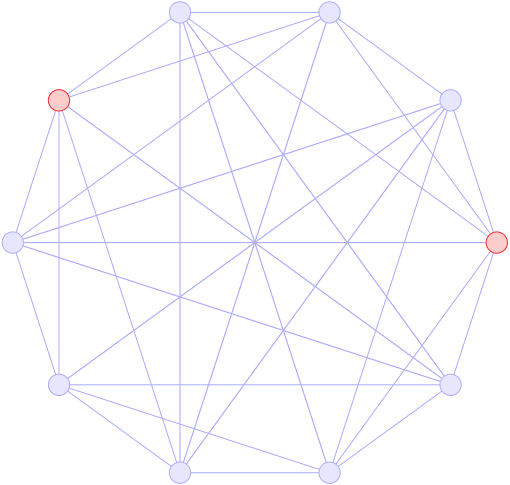

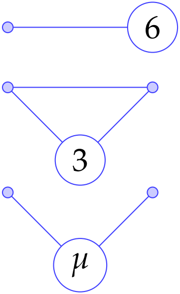

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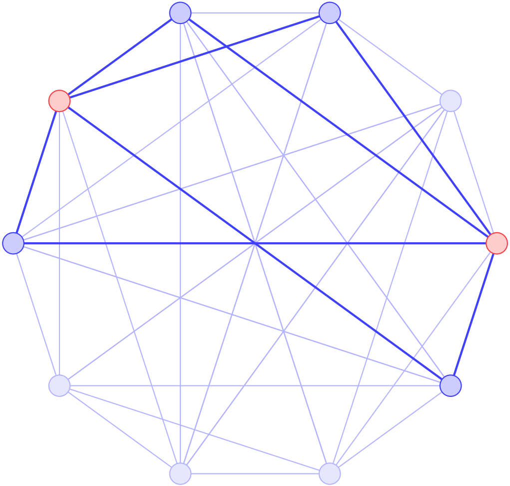

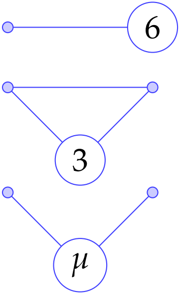

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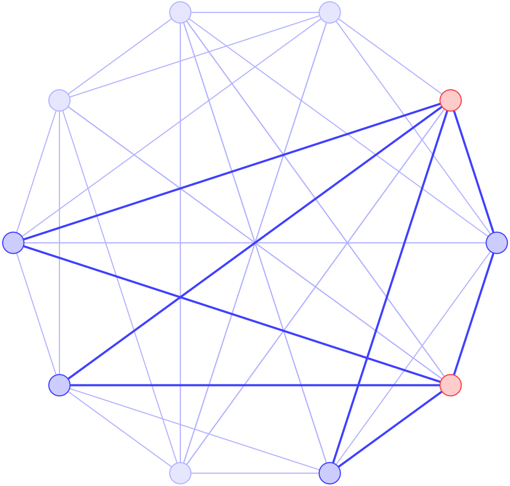

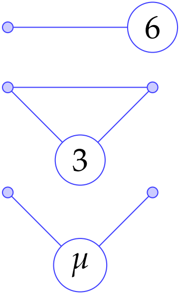

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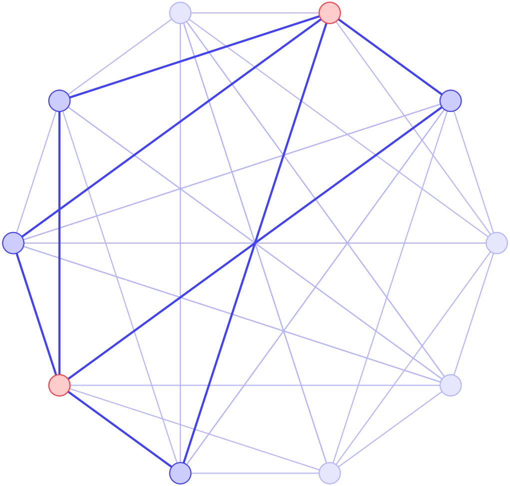

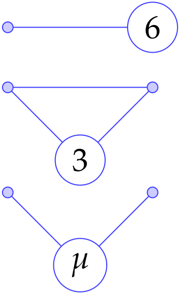

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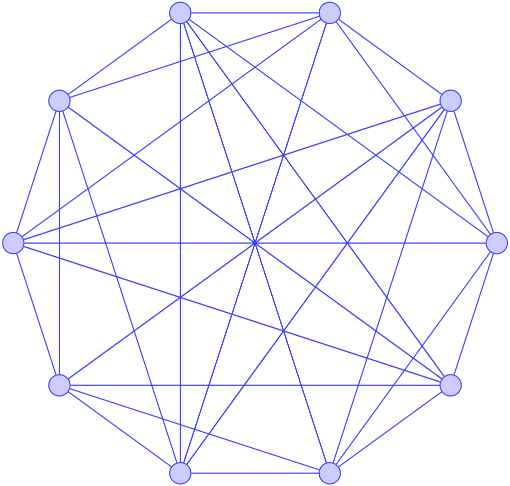

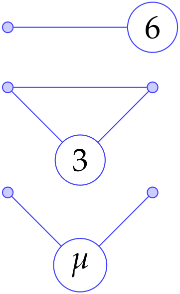

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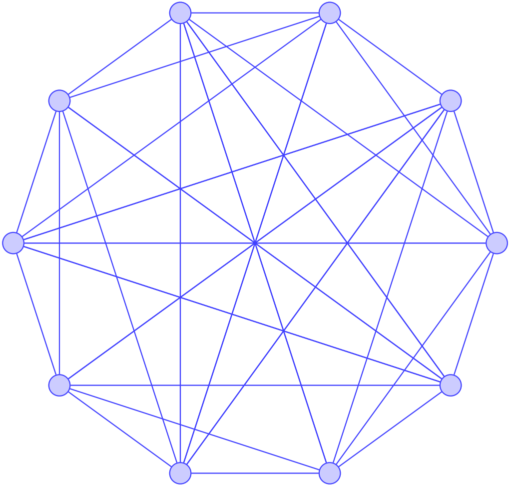

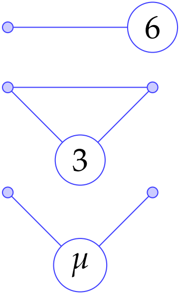

$\mu = 4$

strongly regular
$\mathrm{srg}(10, 6, 3, 4)$

Question (Neumaier 1981)

Is every edge-regular graph with a regular clique strongly regular?

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Answer (GG and Koolen 2018)

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No.

Gary — magma.exe ◂ magma — 94×34

```
Graph
Vertex  Neighbours

1       8 9 14 15 18 19 22 24 27 ;
2       8 9 10 16 19 20 23 25 28 ;
3       9 10 11 17 20 21 22 24 26 ;
4       10 11 12 15 18 21 23 25 27 ;
5       11 12 13 15 16 19 24 26 28 ;
6       12 13 14 16 17 20 22 25 27 ;
7       8 13 14 17 18 21 23 26 28 ;
8       1 2 7 15 17 20 22 25 26 ;
9       1 2 3 16 18 21 23 26 27 ;
10      2 3 4 15 17 19 24 27 28 ;
11      3 4 5 16 18 20 22 25 28 ;
12      4 5 6 17 19 21 22 23 26 ;
13      5 6 7 15 18 20 23 24 27 ;
14      1 6 7 16 19 21 24 25 28 ;
15      1 4 5 8 10 13 22 23 28 ;
16      2 5 6 9 11 14 22 23 24 ;
17      3 6 7 8 10 12 23 24 25 ;
18      1 4 7 9 11 13 24 25 26 ;
19      1 2 5 10 12 14 25 26 27 ;
20      2 3 6 8 11 13 26 27 28 ;
21      3 4 7 9 12 14 22 27 28 ;
22      1 3 6 8 11 12 15 16 21 ;
23      2 4 7 9 12 13 15 16 17 ;
24      1 3 5 10 13 14 16 17 18 ;
25      2 4 6 8 11 14 17 18 19 ;
26      3 5 7 8 9 12 18 19 20 ;
27      1 4 6 9 10 13 19 20 21 ;
28      2 5 7 10 11 14 15 20 21 ;
```

Question (Neumaier 1981)

Is every edge-regular graph with a regular clique strongly regular?

Answer (GG and Koolen 2018)

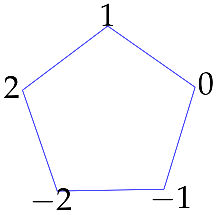
No. There exist infinitely many non-strongly-regular, edge-regular vertex-transitive graphs with regular cliques.

Cayley graphs

- Let G be an (additive) group and $S \subseteq G$ a generating subset, i.e., $G = \langle S \rangle$.
- The **Cayley graph** $\mathrm{Cay}(G, S)$ has vertex set G and arc set
$$\{(g, g+s) : g \in G \text{ and } s \in S\}.$$

Example

$\Gamma = \mathrm{Cay}(\mathbb{Z}_5, S)$ Generating set $S = \{-1, 1\}$



An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$

Generating set S

$$\begin{array}{ll} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is **edge-regular** $(28, 9, 2)$:

Generating set S

$$\begin{array}{cc} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{array}$$

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Generating set S

$$\boxed{\begin{matrix} (01,0) & (01,\pm 1) \\ (10,0) & (10,\pm 2) \\ (11,0) & (11,\pm 3) \end{matrix}}$$

$(00, 0)$

An example

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Generating set S

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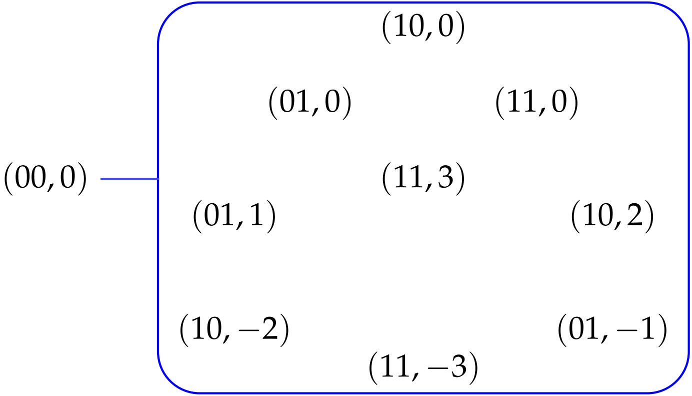

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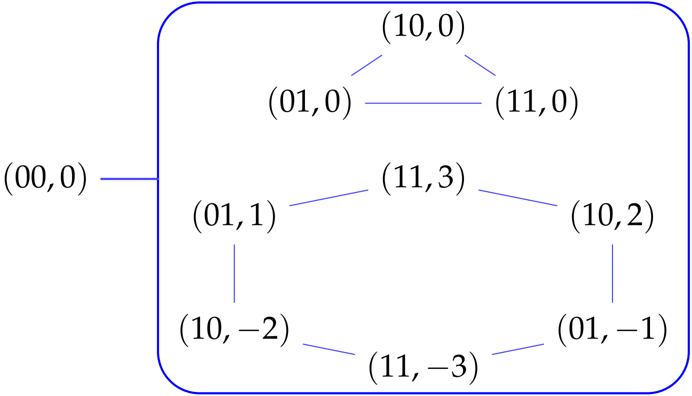

An example

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-**regular** 4-**clique**:

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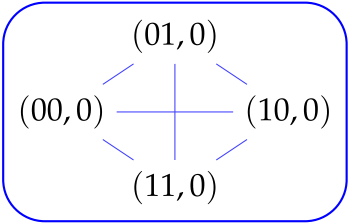

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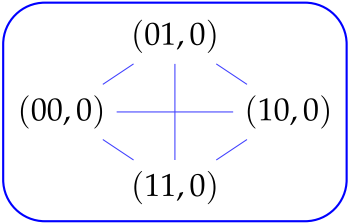

$(a, b) \quad b \neq 0$

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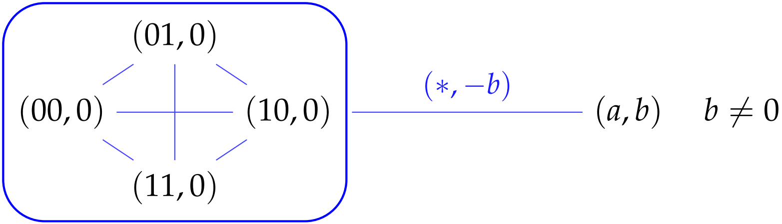

An example

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- Γ is edge-regular $(28, 9, 2)$;
- Γ has a 1-regular 4-clique;

- Γ is **not strongly regular**:

Generating set S

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Generating set S

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$(11, 0)$	$(11, \pm 3)$

- Γ is **not strongly regular**:

$(00, 0)$ —

	$(11, 0)$
$(01, 0)$	$(01, 1)$
$(11, 3)$	$(10, 0)$
$(10, -2)$	$(01, -1)$
$(11, -3)$	$(10, 2)$

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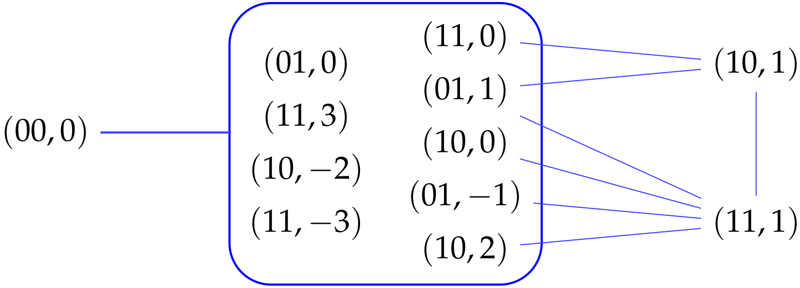

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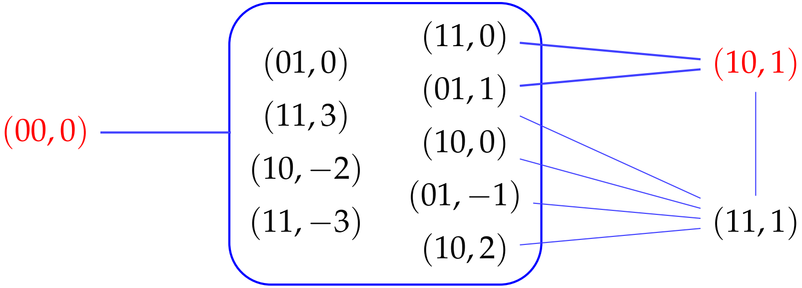

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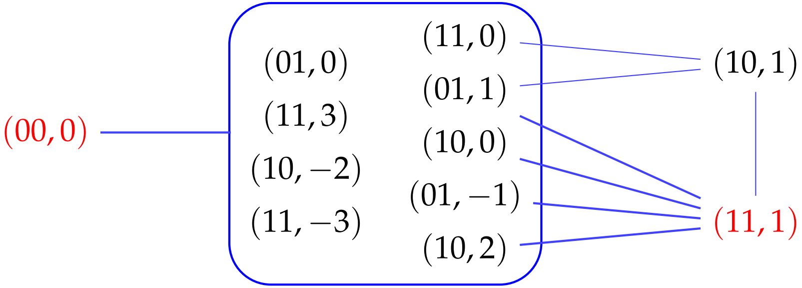

A general construction

- $\Gamma = \mathrm{Cay}(\mathbb{Z}_2^2 \oplus \mathbb{Z}_7, S)$
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Generating set S

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$$\Downarrow$$

- Bijection $\pi : (\mathbb{Z}_2^2)^* \to \mathbb{Z}_3$
- Primitive element ρ of $\mathbb{F}_q$
- $\Gamma = \mathrm{Cay}(\mathbb{Z}_l \oplus \mathbb{Z}_2^2 \oplus \mathbb{F}_q, S(\pi))$

Generating set $S(\pi)$

$$(g, 0) \mid g \in (\mathbb{Z}_l \oplus \mathbb{Z}_2^2)^*$$

$$\forall z \in (\mathbb{Z}_2^2)^*$$
$$(0, z, \rho^j) \mid j \equiv \pi(z) \pmod 3$$

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$$\forall z \in (\mathbb{Z}_2^2)^*$$
$$(0,z,\rho^j) \mid j \equiv \pi(z) \pmod 3$$

- Γ is undirected when $q \equiv 1 \pmod 6$;
- Γ has a (spread of) 1-regular clique(s);
- When $q \equiv 1 \pmod 6$, the graph induced on the neighbourhood $\Gamma(v)$ has valencies $4l - 2$ and $2c$, where c equals the 3rd cyclotomic number $c_q^3(1,2)$.

General construction/infinite family

- Generalise: $\mathbb{Z}_2^2 \oplus \mathbb{Z}_7$ to $\mathbb{Z}_{(c+1)/2} \oplus \mathbb{Z}_2^2 \oplus \mathbb{F}_q$.

- Works for $q \equiv 1 \pmod 6$ such that the 3rd cyclotomic number $c = c_q^3(1,2)$ is odd.

- Then there exists an $\mathrm{erg}(2(c+1)q, 2c+q, 2c)$ having a 1-regular clique of order $2c+2$.

- Take $p \equiv 1 \pmod 3$ a prime s.t. $2 \not\equiv x^3 \pmod p$. Then there exist a such that $c_{p^a}^3(1,2)$ is odd.

Examples in the wild

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 11, стр. 268–310 (2014)

УДК 519.17
MSC 05C

КЭЛИ-ДЕЗА ГРАФЫ, ИМЕЮЩИЕ МЕНЕЕ 60 ВЕРШИН

С.В. ГОРЯИНОВ, Л.В. ШАЛАГИНОВ

Abstract. Deza graph, which is the Cayley graph is called the Cayley-Deza graph. The paper describes all non-isomorphic Cayley-Deza graphs of diameter 2 having less than 60 vertices.

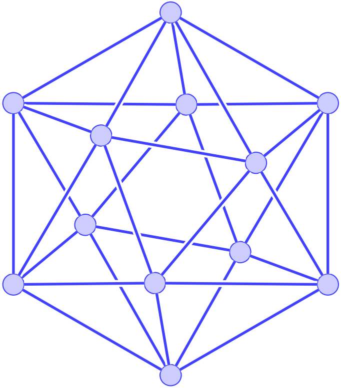
Keywords: Deza graph, Cayley graph, graph isomorphism, automorphism group.

1. Введение

В этой статье мы начинаем изучение графов Деза, которые являются графами Кэли. Графы Деза принято рассматривать как обобщение сильно регулярных графов. В ряде исследований было выяснено, что графы Деза наследуют некоторые свойства сильно регулярных графов. Например, в [1] показано, что вершинная связность графа Деза, полученного из сильно регулярного графа с помощью инволюции, совпадает с валетностью.

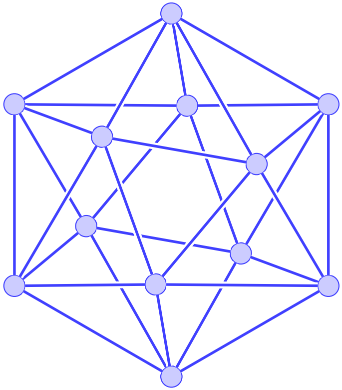
- Four $\mathrm{erg}(24, 8, 2)$ graphs with a 1-regular clique;
- Brought to our attention by Sasha Gavrilyuk

Another construction



start with:
2-antipodal $(12, 5, 2)$-erg
diameter 3

Another construction



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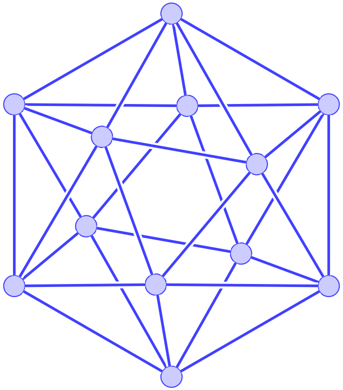

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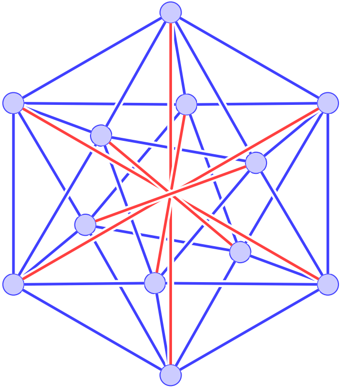

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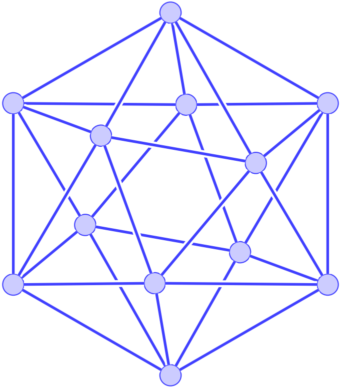

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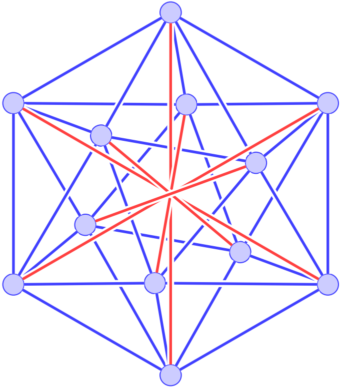

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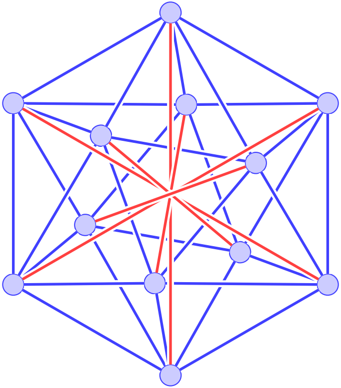

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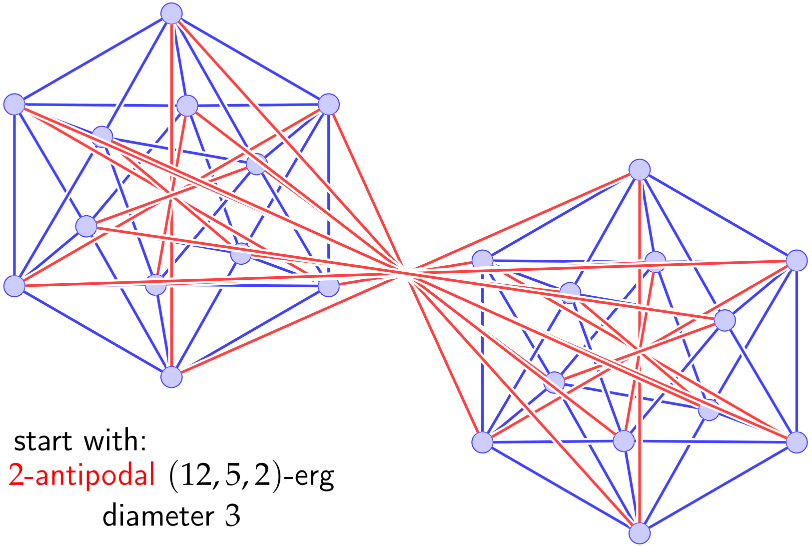

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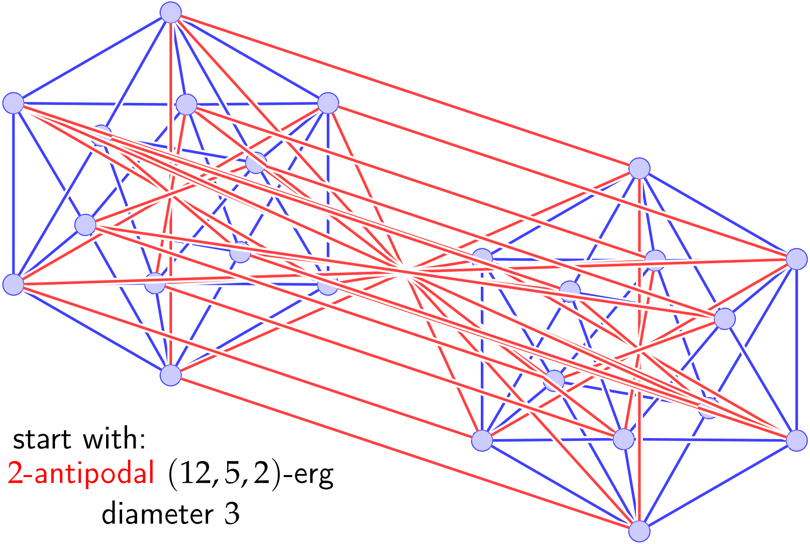

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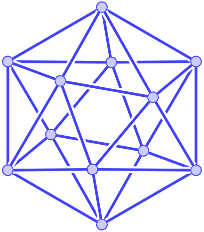

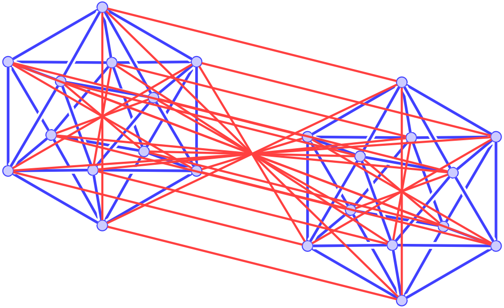

2-antipodal $(12, 5, 2)$-erg
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$\rightarrow$

$(24, 8, 2)$-erg with
1-regular 4-cliques
not strongly regular

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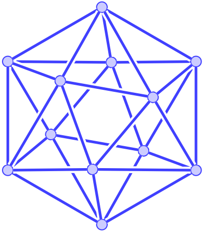

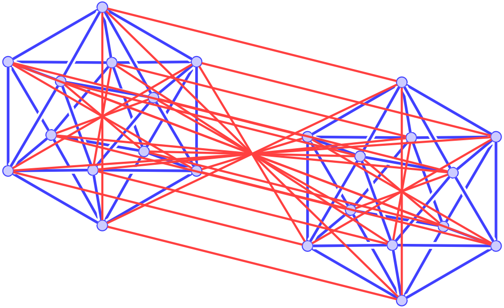

2-antipodal $(12, 5, 2)$-erg
diameter 3

$\rightarrow$

$(24, 8, 2)$-erg with
1-regular 4-cliques
not strongly regular

a-antipodal (v, k, λ)-erg
diameter 3

$\rightarrow$

$(\frac{v(\lambda+2)}{a}, k+\lambda+1, \lambda)$-erg with
1-regular $(\lambda+2)$-cliques
not strongly regular

Remarks

- A non-strongly-regular, edge-regular graph with a regular clique is called a **Neumaier graph**.
- a-antipodal $\mathrm{erg}(v,k,\lambda)$ to $\mathrm{erg}(v(\lambda+2)/a, k+\lambda+1, \lambda)$.
  - Produce Neumaier graphs using constructions of antipodal edge-regular graphs of diameter 3;
- Evans, Goryainov, and Panasenko (2018+): Smallest Neumaier graph is $\mathrm{erg}(16,9,4)$.
- Evans, Goryainov, and Panasenko (2018+): Neumaier graphs with 2-regular cliques.

Problems

- $\exists$ Neumaier graphs with 3-regular cliques?
- $\exists$ Neumaier graphs with diameter $\geqslant 3$?
- Spectral properties of Neumaier graphs

Thank you for your attention

Further reading:

G. R. W. Greaves and J. H. Koolen, *Edge-regular graphs with regular cliques*, European J. Combin. **71** (2018), pp. 194–201.