

# Quantum walks and algebraic graph theory

Krystal Guo

Université Libre de Bruxelles

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$U(t)$  is unitary and symmetric matrix.

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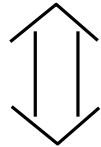
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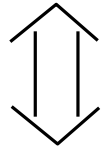
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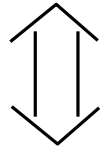


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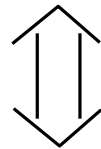
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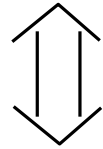
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We can take a spectral relaxation of this property.

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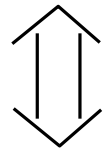
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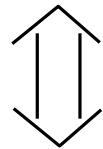
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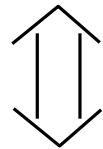
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To study quantum walks, we need another concept:

Vertices  $u$  and  $v$  are **strongly cospectral**



there exists a **orthogonal** matrix  $Q$  such that

- (a)  $Q$  is a polynomial in  $A$  with rational entries;
- (b)  $Qe_u = e_v$ ;
- (c)  $Q^2 = I$ .



# Strong Cospectrality in Quantum Walks

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(Godsil 2018) Two columns of  $\widehat{M}$  are equal if and only if the corresponding vertices are strongly cospectral.

Perfect state transfer on distance-regular graphs and association schemes. G. Coutinho, C. Godsil, K. Guo, F. Vanhove. Linear Algebra and its Applications 449 (2015) P108-130.

State transfer in strongly regular graphs with an edge perturbation. C. Godsil, K. Guo, M. Kempton and G. Lippner.

A new perspective on the average mixing matrix. G. Coutinho, C. Godsil, K. Guo and H. Zhan.