Quantum walks and algebraic graph theory

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$U(t)$ is unitary and symmetric matrix.

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(a) Q is a polynomial in A with rational entries;

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Strong Cospectrality in Quantum Walks

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(Godsil 2018) Two columns of $\widehat{M}$ are equal if and only if the corresponding vertices are strongly cospectral.

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