Quantum walks and algebraic graph theory

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Continuous-time quantum walk

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Continuous-time quantum walk

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$$= \begin{pmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{pmatrix}$$
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$U(t)$ is unitary and symmetric matrix.
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perfect state transfer between $u$ and $v$

exists $\tau$, \[ U(\tau) = \gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] \[ \|\gamma\| = 1 \]

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\[ \begin{array}{c}
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We can take a spectral relaxation of this property.

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We can take a spectral relaxation of this property.

Vertices $u$ and $v$ are cospectral

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To study quantum walks, we need another concept:

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"Algebraically" symmetric

To study quantum walks, we need another concept:

Vertices $u$ and $v$ are strongly cospectral

there exists an orthogonal matrix $Q$ such that

(a) $Q$ is a polynomial in $A$ with rational entries;
(b) $Qe_u = e_v$;
(c) $Q^2 = I$. 
Strong Cospectrality in Quantum Walks

Perfect state transfer between $u$ and $v$

Average mixing matrix
Strong Cospectrality in Quantum Walks

perfect state transfer between \( u \) and \( v \)

\[ \rightarrow \quad u \text{ and } v \text{ are strongly cospectral} \]

average mixing matrix
Strong Cospectrality in Quantum Walks

perfect state transfer between $u$ and $v$

$\implies$ $u$ and $v$ are strongly cospectral

average mixing matrix

(Godsil 2018) Two columns of $\hat{M}$ are equal if and only if the corresponding vertices are strongly cospectral.

State transfer in strongly regular graphs with an edge perturbation. C. Godsil, K. Guo, M. Kempton and G. Lippner.