On p-valenced association schemes whose thin residue has valency p^2

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Definition (see [BI], [BCN], [EP], [Z1])

An association scheme (X, R) is called an **extension** of (Y, S) by (Z, T) if there exists an equivalence E (closed subset, imprimitive block) of R one of whose equivalence classes induces (Y, S) and whose quotient is isomorphic to (Z, T).

Notice

A finite group G is identified with an association scheme $(G, \{\bar{g} \mid g \in G\})$ where $\bar{g} := \{(a, b) \in G \times G \mid ab^{-1} = g\}$.

Examples

$$C_3 \simeq \{I, C, C^2\} := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$$C_3 \wr C_3 = \{I \otimes X \mid X \in C_3\} \cup \{Y \otimes J \mid Y \in C_3 \setminus \{I\}\}$$

$$\left(\begin{array}{ccc|ccc|ccc} 0 & 1 & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\ 2 & 0 & 1 & 3 & 3 & 3 & 4 & 4 & 4 \\ 1 & 2 & 0 & 3 & 3 & 3 & 4 & 4 & 4 \\ \hline 4 & 4 & 4 & 0 & 1 & 2 & 3 & 3 & 3 \\ 4 & 4 & 4 & 2 & 0 & 1 & 3 & 3 & 3 \\ 4 & 4 & 4 & 1 & 2 & 0 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 4 & 4 & 4 & 0 & 1 & 2 \\ 3 & 3 & 3 & 4 & 4 & 4 & 2 & 0 & 1 \\ 3 & 3 & 3 & 4 & 4 & 4 & 1 & 2 & 0 \end{array}\right).$$

An extension of $C_3 \wr C_3$ by C_3

$$\left(\begin{array}{ccc|ccc|ccc} O & O & O & I & C & C & O & O & O \\ O & O & O & C & I & C & O & O & O \\ O & O & O & C & C & I & O & O & O \\ \hline O & O & O & O & O & O & I & C & C \\ O & O & O & O & O & O & C & I & C \\ O & O & O & O & O & O & C & C & I \\ \hline I & C & C & O & O & O & O & O & O \\ C & I & C & O & O & O & O & O & O \\ C & C & I & O & O & O & O & O & O \end{array}\right).$$

$$\begin{pmatrix} I & C & C \\ C & I & C \\ C & C & I \end{pmatrix} \begin{pmatrix} I & C^2 & C^2 \\ C^2 & I & C^2 \\ C^2 & C^2 & I \end{pmatrix} = \begin{pmatrix} 3I & J & J \\ J & 3I & J \\ J & J & 3I \end{pmatrix},$$

$$\begin{pmatrix} I & C & C \\ C & I & C \\ C & C & I \end{pmatrix} \begin{pmatrix} I & C & C \\ C & I & C \\ C & C & I \end{pmatrix} = (I + 2C^2) \begin{pmatrix} I & C^2 & C^2 \\ C^2 & I & C^2 \\ C^2 & C^2 & I \end{pmatrix}.$$

Characterization by the thin residue

Definition

Every association scheme has the smallest equivalence whose quotient is a group, called the **thin residue**.

[Z0]

For each association scheme (X, S), if $\mathbf{O}^{\theta}(S) \simeq C_n$,
then (X, S) is schurian.

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Q3. How many such $\{1, 3\}$-schemes such that each relation out of the thin residue has valency 3?
A3. Finite. I will show you the reason why in this talk.

Assumption

(X, S): an association scheme with $\mathbf{O}^{\theta}(S) \simeq C_p \wr C_p$ where p is a prime
Assume that $n_s = p$ for each $s \in S \setminus \mathbf{O}^{\theta}(S)$.
$(X, \tilde{S})$: the thin residue extension of (X, S), (see [EP], [MP]) i.e.,
the smallest coherent configuration containing S and $x\mathbf{O}^{\theta}(S) \times x\mathbf{O}^{\theta}(S)$ with $x \in X$
$X_1, X_2, \ldots, X_m$: the distinct fibers of $(X, \tilde{S})$
$\tilde{S}_{ij} := \{s \in \tilde{S} \mid s \subseteq X_i \times X_j\}$ for $i, j = 1, 2, \ldots, m$

Main Result

Lemma 1

For each $s \in \tilde{S}_{ij}$ with $i \neq j$ the adjacency $s \cap (X_i \times X_j)$ induces a complex Hadamard matrix $H(s)$ of Butson type (p, p).

Lemma 2

For all $s_1 \in \tilde{S}_{ij}$, $s_2 \in \tilde{S}_{jk}$ and $s_3 \in \tilde{S}_{ik}$ with $i \neq j$, $j \neq k$ and $i \neq k$ we have $H(s_1)H(s_2) = \alpha H(s_3)$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = \sqrt{p}$.

Lemma 3

The rows of I, $\frac{1}{\sqrt{p}}H(s_{i1})$ with $i = 2, 3, \ldots, m$ forms a mutually unbiased bases for $\mathbb{C}^p$ where $s_{i1} \in \tilde{S}_{i1}$. Thus, $m \leq p + 1$ (see [BBRF]).

Theorem (H, W. Abbas)

We have $|X| \leq p^2(p+1)$.

Why $\mathbf{O}^\theta(S) \simeq C_p \wr C_p$?

We say that (X, S) is a p-scheme if $|s|$ is a power of p for each $s \in S$ where $|s| = |X|n_s$.
Each p-scheme has a series $\{S_i\}_{i=0}^k$ of equivalences such that $\{1_X\} = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = S$ and $S_{i+1}/\!\!/S_i \simeq C_p$ for $i = 0, 1, \ldots, k-1$.
(1) If $|X| = p$, then $(X, S) \simeq C_p$;
(2) If $|X| = p^2$, then $(X, S) \simeq C_{p^2}$, $C_p \times C_p$ or $C_p \wr C_p$;
(3) If $|X| = p^3$, then only the case of $p = 2, 3$ are classified.

(4) If $\mathbf{O}^\theta(S)$ is cyclic, then (X, S) is schurian (see [HZ]);
(5) If $\mathbf{O}^\theta(S) \simeq C_p \times C_p$, then [BH] and [CHK] show some constructions of non-schurian p-schemes;
(6) If $\mathbf{O}^\theta(S) \simeq C_p \wr C_p$, then the problem is reduced to the case of $|X| = p^3$ by our main result.

On the case $\mathbf{O}^\theta(S) \simeq C_p \times C_p$

We consider (X, S) with $\mathbf{O}^\theta(S) \simeq C_p \times C_p$.
Suppose that $n_s = p$ for each $s \in S \setminus \mathbf{O}^\theta(S)$.

Theorem([CHK])

We have $|X| \leq p^2(p^2 + p + 1)$.

Let $X_1, \ldots, X_m$ be the distinct geometrical cosets of $\mathbf{O}^\theta(S)$ and
$\mathcal{P} := \{X_1, X_2, \ldots, X_m\}$, $\mathcal{L} := \{L_i(M) \mid 1 \leq i \leq m, 1 < M < \mathbf{O}^\theta(S)\}$
where $L_i(M) := \{X_i\} \cup \{X_j \mid \forall s \in S; s \cap (X_i \cap X_j) \neq \emptyset, ss^* = M\}$.
Then $(\mathcal{P}, \mathcal{L})$ is a linear space.

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Thank you for your attention.