Maximal Commutative Schur rings

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Schur rings

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S. Humphries, K. Johnson, A. Misseldine (BY)  Maximal Commutative Schur rings  July 3, 2018 2 / 20
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A Schur-ring (or S-ring) over a group $G$ is a sub-ring $\mathcal{S}$ of $\mathbb{C}G$ that is constructed from a partition $\{\Gamma_1, \Gamma_2, \ldots\}$ of the elements of $G$ by finite sets with $\Gamma_1 = \{id\}$, satisfying:
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2. if $1 \leq i, j \leq m$, then

$$\overline{\Gamma_i \Gamma_j} = \sum_{k=1}^{m} \lambda_{ijk} \overline{\Gamma_k},$$

where $\lambda_{ijk} \in \mathbb{Z}_{\geq 0}$ for all $i, j, k$. 
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where $\lambda_{ijk} \in \mathbb{Z}_{\geq 0}$ for all $i, j, k$.

The $\Gamma_i$ are called the principal sets of the S-ring.
The maximal dimension of a commutative $S$-ring over $G$ is

$$s_G := \sum_{i=1}^{r} d_i,$$

where $d_1, \ldots, d_r$ are the irreducible character degrees of $G$. 
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We say that the finite group $G$ is $S$-realizable if there is a commutative $S$-ring $\mathcal{S}$ over $G$ of dimension $s_G$. We will also say that $\mathcal{S}$ realizes $s_G$. 
Some motivation

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Use: diagonalization.
We give examples of families of finite groups that are $S$-realizable. For example we have:

**Theorem**

The groups $\text{SL}(2, 2^n)$, $n \geq 1$, are $S$-realizable.

**Metacyclic groups:** have a cyclic normal subgroup $N$ such that $G/N$ is cyclic.

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Every finite metacyclic group $G$ is $S$-realizable.
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Theorem

Every finite metacyclic group $G$ is $S$-realizable.
Amitsur determines all groups whose irreducible characters have degrees bounded by 2. These are:

1. Abelian groups.
2. Groups $G$ with a normal abelian subgroup of index 2.
3. Groups $G$ such that $G/Z(G) \cong \mathbb{Z}/2\mathbb{Z}$.

We call these Amitsur groups (of types (1), (2), (3)).

Theorem
(i) The Amitsur groups of type (1), (2) are $S$-realizable.
(ii) The Amitsur groups of type (3), but not of type (2), are not $S$-realizable.

We generalize the above result (to the situation where 2 is replaced by a prime $p > 2$).
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Examples for small groups

If $G$ is a group and $H$ is a finite subgroup, then the orbits for the conjugation action of $H$ on $G$ partition $G$. These orbits determine an $S$-ring that we denote by $\mathcal{S}(G, H)$. 

1. For $G = A_4$ we take $H = \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$. Then $\mathcal{S}(G, H)$ realizes $s_G$.

2. For $G = S_4$ we take $H = \langle (1, 2, 3) \rangle$. Then $\mathcal{S}(G, H)$ realizes $s_G$.

3. For $G = A_5$ we take $H = \langle (1, 2, 3, 4, 5) \rangle$. Then $\mathcal{S}(G, H)$ realizes $s_G$.

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S. Humphries, K. Johnson, A. Misseldine (BYU)  Maximal Commutative Schur rings  July 3, 2018  7 / 20
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A computer calculation shows that for all groups $G$ of order less than 54 there is a subgroup $H$ such that the $S$-ring $\mathcal{S}(G, H)$ realizes $s_G$. Examples:

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We also have:

**Theorem**

Let $G$ be a Frobenius group with Frobenius complement $H$ and abelian Frobenius kernel. Then $G$ is $S$-realizable if and only if $H$ is $S$-realizable.

We note that $S$-realizability is inherited by quotients:

**Theorem**

Let $G$ be a finite group which is $S$-realizable and let $H \trianglelefteq G$. Then $G/H$ is $S$-realizable.
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Group determinant

For a finite group $G = \{g_1 = 1, g_2, \ldots, g_n\}$, and independent commuting variables $x_1, x_2, \ldots, x_n, x_i = x_{g_i}$, the group matrix $X_G$ of $G$ is the $n \times n$ matrix whose rows and columns are indexed by the group elements, where the $(i, j)$ entry is $x_k$ if $g_i g_j^{-1} = g_k$. The group determinant is $\Theta_G = \det X_G$.

**Theorem**

Let $S$ be an $S$-ring on a finite group $G$ and let $\{C_i\}_{i=1}^t$ be the principal sets of $S$. Suppose that each variable $x_g$ in $X_G$ is set equal to the variable $x_{C_i}$, where $g \in C_i$, to obtain the matrix $X_G^S$. Let $\Theta_G^S = \det X_G^S$.

Then $\Theta_G^S$ factors into linear factors (over $\mathbb{C}$) if and only if $S$ is commutative.
\[ \mathcal{S}(G, H) \text{ maximal} \]

**Theorem**

*If* \( \mathcal{S}(G, H) \) *is commutative and realizes* \( s_G \), *then* \( H \) *is abelian.*
Proof: Let $H \leq G$. 
Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of $G$. 
Let $\varphi_1, \ldots, \varphi_s$ be the irreducible characters of $H$. 
Let $E_1, \ldots, E_t$ be the $H$-conjugacy classes.
Proof: Let $H \leq G$.
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Karlof/Wigner: Suppose that $\chi_j|_H = \sum_{i=1}^{s} c_{ij} \varphi_i$. Then $\sum_{i,j} c_{ij}^2 = t.$
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So \( \chi_j(1) = \sum_{i=1}^s c_{ij} \varphi_i(1) \), and

\[
S_G = \sum_{j=1}^r \chi_j(1) = \sum_{i,j} c_{ij} \varphi_i(1).
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$$s_G = \sum_{j=1}^r \chi_j(1) = \sum_{i,j} c_{ij} \varphi_i(1).$$

Thus if the S-ring $\mathcal{S}(G, H)$ is of maximal dimension, then $t = s_G$ and so

$$\sum_{i,j} c_{ij} \varphi_i(1) = s_G = t = \sum_{i,j} c_{ij}^2.$$
Karof: The irreducible $\mathcal{S}(G, H)$-modules have dimension $c_{ij}$. 

So: $\mathcal{S}(G, H)$ is commutative if and only if $c_{ij} = 0, 1$.

If $\mathcal{S}(G, H)$ is maximal commutative, then $c_{ij} \in \{0, 1\}$ and also \[
\sum_{i,j} c_{ij} \varphi_i(1) = s_G = t = \sum_{i,j} c_{ij} \varphi_i(2).
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But $\varphi_i(1) \geq 1$ and so $\varphi_i(1) = 1$ for all $\varphi_i$.

Theorem: If $\mathcal{S}(G, H)$ is commutative of maximal dimension, then $H$ is abelian.
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Gelfand pairs

1. $(G, H), H \leq G$, is a Gelfand pair if $\langle \chi|_H, 1 \rangle \leq 1$ for all irreducible characters $\chi$ of $G$
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Nice fact: \((G, H), H \leq G,\) is a strong Gelfand pair if and only if \(S(G, H)\) is commutative.
Symmetric groups

We note:

$(S_3, \langle (1, 2) \rangle)$ is a strong Gelfand pair realizing $s_G$.
$(S_4, \langle (1, 2, 3) \rangle)$ is a strong Gelfand pair realizing $s_G$.
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**Proposition**

*If \(n \geq 6\) and \(H \leq S_n\) is an abelian subgroup, then \((G, H)\) is not a Gelfand pair.*

*In particular, \(S_n, n \geq 6\), cannot realize \(s_G\) using an \(S\)-ring of the form \(\mathcal{S}(S_n, H)\).*
Symmetric groups

Proof:
Let $V_\pi = \text{Span}(v_1, \ldots, v_n) = V_{(n-1,1)} \oplus V_{(n)}$ be the permutation module.
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2. If $k = 1$ : A transitive abelian subgroup of $S_n$ is regular. Then the action of $H \leq S_n$ on $V_\pi \otimes V_\pi$ has at least $n$ orbits ($v_1 \otimes v_i, i \geq 1$)
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3. If \( k = 2 \), then \( V_{(n-1,1)} \otimes V_{(n-1,1)} \) has at least \( n - 1 \) orbits.

Use:

\[
V_\pi \otimes V_\pi = 2V_{(n)} \oplus 3V_{(n-1,1)} \oplus V_{(n-2,1,1)} \oplus V_{(n-2,2)}.
\]
Question 1: What is the largest dimension of a commutative S-ring over $S_n$. 

Conjecture: when $H = S_n - 2 \times S_2$, $n > 5$. 

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Infinite groups

Question 1: What is the largest dimension of a commutative S-ring over $S_n$.

Question 2: What is the largest dimension of a commutative S-ring of the form $\mathcal{G}(S_n, H)$. 

Conjecture: when $H = S_n - 2 \times S_2$, $n > 5$. 

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Infinite groups

(with Bastian, Brewer, Misseldine and Thompson)

Theorem

Any Schur ring over \( \mathbb{Z} \) or \( \mathbb{Z} \times C_2 \) is the set of orbits of a finite subgroup of automorphisms of the group or a wedge product.

More generally: a group is locally infinite-cyclic if any finite non-trivial set generates an infinite cyclic subgroup. e.g. subgroups of \( \mathbb{Q} \).
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**Theorem**

Let \( G \) be locally infinite-cyclic. Any Schur ring over \( G \) or \( G \times C_2 \) is generated as the orbits of a finite subgroup of automorphisms of the group or a wedge product.
Let $F_n$ be a free group of rank $n$ and let $L_k = L_{n,k} = \{x : x \in F_n, |x| = k\}$.

**Theorem**

The $L_k$ are the principle sets of a Schur ring over $F_n$. Further, this S-ring is commutative.
Infinite groups - free products

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Then a type will be a sequence $s = [s_1, s_2, \ldots, s_h]$ where $s_i \in \{G_i\}_i$ and $s_i \neq s_{i+1}$.
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Let $\mathcal{L}_s$ be the set of all elements of the form $w = w_1 w_2 \cdots w_h$ where $w_i$ is a non-trivial principal element of $O_{s_i}$. Then:

**Theorem**

The set $\bigcup_s \mathcal{L}_s$, where $s$ is a type, is a set of principle elements for a Schur ring over $G$. 