Maximal Commutative Schur rings

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Schur rings

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(2) if $1 \le i, j \le m$, then

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where $\lambda_{ijk} \in \mathbb{Z}^{\ge 0}$ for all i, j, k.

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The Γ_i are called the *principal sets* of the S-ring.

Basic result

The maximal dimension of a commutative S-ring over G is

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We say that the finite group G is *S-realizable* if there is a commutative S-ring $\mathfrak{S}$ over G of dimension s_G.
We will also say that $\mathfrak{S}$ *realizes* s_G.

Some motivation

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Use: diagonalization.

Result for $SL(2, 2^n)$ and metacyclic groups

We give examples of families of finite groups that are S-realizable. For example we have:

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Every finite metacyclic group G is S-realizable.

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We generalize the above result (to the situation where 2 is replaced by a prime $p > 2$).

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If G is a group and H is a finite subgroup, then the orbits for the conjugation action of H on G partition G.
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3. For $G = A_5$ we take $H = \langle (1,2,3,4,5) \rangle$. Then $\mathfrak{S}(G, H)$ realizes s_G.

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Frobenius groups and quotients

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We note that S-realizability is inherited by quotients:

Theorem

Let G be a finite group which is S-realizable and let $H \trianglelefteq G$. Then G/H is S-realizable.

Group determinant

For a finite group $G = \{g_1 = 1, g_2, \ldots, g_n\}$, and independent commuting variables $x_1, x_2, \ldots, x_n, x_i = x_{g_i}$, the *group matrix* X_G of G is the $n \times n$ matrix whose rows and columns are indexed by the group elements, where the (i, j) entry is x_k if $g_i g_j^{-1} = g_k$. The *group determinant* is $\Theta_G = \det X_G$.

Theorem

Let $\mathfrak{S}$ be an S-ring on a finite group G and let $\{C_i\}_{i=1}^t$ be the principal sets of $\mathfrak{S}$. Suppose that each variable x_g in X_G is set equal to the variable x_{C_i}, where $g \in C_i$, to obtain the matrix $X_G^{\mathfrak{S}}$. Let $\Theta_G^{\mathfrak{S}} = \det X_G^{\mathfrak{S}}$.

Then $\Theta_G^{\mathfrak{S}}$ factors into linear factors (over $\mathbb{C}$) if and only if $\mathfrak{S}$ is commutative.

$\mathfrak{S}(G, H)$ maximal

Theorem

If $\mathfrak{S}(G, H)$ is commutative and realizes s_G, then H is abelian.

$\mathfrak{S}(G, H)$ maximal

Proof: Let $H \leq G$.
Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G.
Let $\varphi_1, \ldots, \varphi_s$ be the irreducible characters of H.
Let $E_1, \ldots, E_t$ be the H-conjugacy classes.

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So $\chi_j(1) = \sum_{i=1}^{s} c_{ij}\varphi_i(1)$, and

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Thus if the S-ring $\mathfrak{S}(G, H)$ is of maximal dimension, then $t = s_G$ and so

$$\sum_{i,j} c_{ij}\varphi_i(1) = s_G = t = \sum_{i,j} c_{ij}^2.$$

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Theorem

If $\mathfrak{S}(G, H)$ is commutative of maximal dimension, then H is abelian.

Gelfand pairs

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Nice fact: $(G, H), H \leq G$, is a strong Gelfand pair if and only if $\mathfrak{S}(G, H)$ is commutative.

Symmetric groups

We note:
$(S_3, \langle (1,2) \rangle)$ is a strong Gelfand pair realizing s_G.
$(S_4, \langle (1,2,3) \rangle)$ is a strong Gelfand pair realizing s_G.
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Proposition

If $n \geq 6$ and $H \leq S_n$ is an abelian subgroup, then (G, H) is not a Gelfand pair.
In particular, $S_n, n \geq 6$, cannot realize s_G using an S-ring of the form $\mathfrak{S}(S_n, H)$.

Proof:
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Use:

$$V_\pi \otimes V_\pi = 2V_{(n)} \oplus 3V_{(n-1,1)} \oplus V_{(n-2,1,1)} \oplus V_{(n-2,2)}.$$

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Conjecture: when $H = S_{n-2} \times S_2, n > 5$.

Infinite groups

(with Bastian, Brewer, Misseldine and Thompson)

Theorem

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More generally: a group is locally infinite-cyclic if any finite non-trivial set generates an infinite cyclic subgroup. e.g. subgroups of $\mathbb{Q}$.

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Theorem

Let G be locally infinite-cyclic. Any Schur ring over G or $G \times \mathcal{C}_2$ is generated as the orbits of a finite subgroup of automorphisms of the group or a wedge product.

Infinite groups - constructions

Let F_n be a free group of rank n and let $L_k = L_{n,k} = \{x : x \in F_n, |x| = k\}$.

Theorem

The L_k are the principle sets of a Schur ring over F_n. Further, this S-ring is commutative.

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Let $\mathcal{L}_s$ be the set of all elements of the form $w = w_1 w_2 \cdots w_h$ where w_i is a non-trivial principal element of O_{s_i}. Then:

Theorem

The set $\cup_s \mathcal{L}_s$, where s is a type, is a set of principle elements for a Schur ring over G.

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