No. 1

Date

Toward the classification of (P and Q)-polynomial schemes

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$\Gamma = (X, R)$ finite connected graph
pts edges
undirected
no loops, no mult edges

$$X \times X \supset R_i = \{ (x,y) \mid \partial(x,y) = i \}$$
distance

A_i : adjacency matrix for R_i

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{otherwise} \end{cases}$$

Γ : <u>P-polynomial</u> if

$A_i = v_i(A)$, $A = A_1$

for some polynomial $v_i(x)$ of degree i, $0 \le i \le D$

$$\mathcal{A} = \langle A_0 = I, A_1, \cdots, A_D \rangle \subseteq \mathrm{Mat}_X(\mathbb{C})$$

<u>Bose-Mesner algebra</u>

(commutative, semi-simple alg.)

$$\mathcal{A} = \langle E_0 = \frac{1}{|X|} J,\ E_1, \cdots, E_D \rangle$$

$$I = E_0 + E_1 + \cdots + E_D$$ primitive idempotents

$$E_i E_j = \delta_{ij} E_i$$

$\mathcal{A}$ is closed under the Hadamard product (entry wise product)

$$J = A_0 + A_1 + \cdots + A_D$$

all one matrix

$$A_i \circ A_j = \delta_{ij} A_i$$

Γ : <u>Q-polynomial</u> if

$$n E_i = v_i^*(n E_1) \quad \text{w.r.t. Hadamard product}$$

for some polynomial $v_i^*(x)$ of degree i, $0 \le i \le D$

$(n = |X|)$

Problem Classify $\Gamma = (X, R)$ with P-polynomial and Q-polynomial property for sufficiently large D.

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le D})$ symmetric association scheme

(P and Q)-polynomial schemes

Bannai's lectures at OSU late in 70's

<u>Examples</u> (Bannai's list, core part)

(1) Johnson scheme $J(N, D)$

Ω : set, $|\Omega| = N$, $\Omega = \overset{D}{\Omega_1} \cup \overset{N-D}{\Omega_2}$

$G = \mathrm{Sym}(\Omega) > H = \mathrm{Sym}(\Omega_1) \times \mathrm{Sym}(\Omega_2)$

$X = G/H \simeq \binom{\Omega}{D}$, $D \leq \frac{N}{2}$

$\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$

$R_i \ni (x, y) \iff |x \cap y| = D - i$ G-orbits on $X \times X$

(2) q-Johnson scheme (Grassmann scheme)

$\Omega = \mathbb{F}_q^N$

$$G = GL(N, q) > H = \begin{array}{c|cc} & D & N-D \\ \hline D & * & * \\ N-D & 0 & * \end{array}$$

$X = G/H \simeq \binom{\Omega}{D}_q$, $D \leq \frac{N}{2}$

$\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$

$R_i \ni (x, y) \iff \dim(x \cap y) = D - i$ G-orbits on $X \times X$

(3) dual polar scheme

$G = B_D(q)$, $C_D(q)$, $D_D(q)$, ${}^2D_{D+1}(q)$

$O(2D+1, q)$, $Sp(2D, q)$, $O^+(2D, q)$, $O^-(2D+2, q)$

${}^2A_{2D}(q)$, ${}^2A_{2D-1}(q)$

$U(2D+1, q)$ $U(2D, q)$

H : the stabilizer of
a maximal totally isotropic subspaces

$X = G/H$: the set of
max. totally isotropic subspaces

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le D})$
G-orbits on $X \times X$

$R_i \ni (x, y)$

$\iff \dim(x \cap y) = D - i$

(4) Hamming scheme $H(D, q)$

$G = \mathrm{Sym}(F) \wr \mathrm{Sym}(\Omega)$, $|F| = q$, $|\Omega| = D$

wreath product

$|G| = (q!)^D \cdot D!$

$H = \mathrm{Sym}(F') \wr \mathrm{Sym}(\Omega)$ $F \supset F'$

$|F'| = q-1$

$X = G/H \simeq F^D$

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le D})$

G-orbits

$R_i \ni (x, y)$

$\Longleftrightarrow \# \{ j \mid x_j \neq y_j \} = i$

Hamming distance

(5) Bilinear forms scheme $Bil_{D\times N}(q)$, $D \le N$

$$GL(D+N, q) > H = \begin{array}{c|c} & \begin{array}{cc} D & N \end{array} \\ \begin{array}{c} D \\ N \end{array} & \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array} = LU$$

Levi decomp.

$L = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, $U = \left(\begin{array}{c|c} 1 & * \\ \hline 0 & 1 \end{array}\right)$

Levi subgp unipotent subgp

$X = H/L \simeq$ the set of $D\times N$ matrices over $\mathbb{F}_q$

$\mathfrak{X} = (X, \{R_i\}_{0\le i \le D})$

H-orbits

$R_i \ni (x, y)$

$\iff$ $\operatorname{rank}(x-y) = i$

(6) Affine schemes

(6-1) Alternating bilinear forms scheme $Alt_n(q)$

dual polar scheme of type $D_n(q)$

$G = O^+(2n, q) > H$ the stabilizer of a max. totally isotropic subspace of $\mathbb{F}_q^{2n}$

$H = \underset{\text{Levi}}{L} \cdot \underset{\text{unipotent}}{U}$ Levi decomposition

$X = H/L \simeq$ the set of alternating bilinear forms on $\mathbb{F}_q^n$

$\mathfrak{X} = (X, \underbrace{\{R_i\}_{0 \le i \le D}}_{H\text{-orbits}})$, $D = \left[\frac{n}{2}\right]$

$R_i \ni (x, y) \iff \operatorname{rank}(x - y) = 2i$

(6-2) Hermitian forms scheme $Her_D(q)$

dual polar scheme of type ${}^2A_{2D-1}(q)$

$G = U(2D, q) > H$ the stabilizer of a max. totally isotropic subspace

$\underset{\text{Levi subgp}}{H = L} \cdot \underset{\text{unipotent subgp}}{U}$ Levi decomposition

$X = H/L \simeq$ the set of hermitian forms on $\mathbb{F}_{q^2}^D$

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le D})$

$R_i \ni (x, y) \iff \operatorname{rank}(x-y) = i$

<u>Examples</u> (Bannai's list, relatives)

(a) bipartite half
or antipodal quotient

(b) 2nd P-poly/Q-poly structure

(c) extended bipartite double
or fusion scheme
or farthest subconstituent

(d) same parameters

dual polar scheme of type $B_D(q)$ }
of type $C_D(q)$ } same parameters

not isomorphic if q is odd

<u>core part</u>

$$
\begin{tikzcd}
 & B_D(q) \arrow[dl, "\text{extended bipartite double}"', "(c)"] \arrow[dr, "\{1,2\}\text{ fusion}", "(c)"'] & \\
D_{D+1}(q) \arrow[rr, "\text{bipartite half}", "(a)"'] \arrow[dr, "\text{farthest subconstituent}"'] & & \frac{1}{2}D_{d+1}(q) \arrow[dl, "\text{farthest subconstituent}"] \\
 & Alt_{D+1}(q) &
\end{tikzcd}
$$

<u>relatives</u>

$$
\begin{tikzcd}
 & C_D(q) \arrow[dl, "(c)"'] \arrow[dr, "(c)"] & \\
Hem_{D+1}(q) \arrow[rr, "(a)", "\text{Relatives Not DT (distance trans.)}"'] \arrow[dr] & & Ust_{\left[\frac{D+1}{2}\right]}(q) \arrow[dl] \\
 & Quad_D(q) &
\end{tikzcd}
$$

Hemmeter scheme $Hem_{D+1}(q)$

Ustimenko scheme $Ust_{\left[\frac{D+1}{2}\right]}(q)$

Quadratic forms scheme $Quad_D(q)$

(d) relatives

twisted q-Johnson scheme $\quad {}^2J_q(N, D)$
$\quad (N = 2D+1)$

(twisted Grassmann scheme
van Dam - Koolen scheme)

$V = \mathbb{F}_q^{2D+1} \quad (D \geq 2)$

$\supset H$ hyper subspace
$\dim H = 2D$

$X_1 = \{ U \subset V \mid \dim U = D+1,\ U \not\subset H \}$

$X_2 = \{ U \subset H \mid \dim U = D-1 \}$

$X = X_1 \cup X_2$

$(U, U') \in X_1 \times X_1$ if $\dim U \cap U' = D$
adjacent $\quad X_1 \times X_2$ if $U \supset U'$
$\quad X_2 \times X_2$ if $\dim U \cap U' = D-2$

$\Gamma = (X, R)$ twisted q-Johnson graph

$\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq D})$ twisted q-Johnson scheme

NOT distance transitive
($\mathrm{Aut}\,\Gamma$ has two orbits on X.)
same parameters with q-Johnson scheme

Doob graph

$$\Gamma = \underbrace{\Delta \times \cdots \times \Delta}^{m} \times H(N, 4)$$

Δ : Shrikhande graph
same parameters with $H(2, 4)$, $q = 4$

Γ : same parameters with $H(N+2m, 4)$

vertex transitive

NOT DT

G : finite group

$G > H$ Gelfand pair

$$(1_H)^G = \chi_0 + \chi_1 + \cdots + \chi_D$$

multiplicity free

$\chi_i \in \mathrm{Irr}(G)$, $\quad \chi_i \neq \chi_j$

$X = G/H$

$V = \mathbb{C}X$ standard module

$$\mathrm{Hom}_G(V, V) \simeq eRe$$

commutative,   where $R = \mathbb{C}G$: group ring

$$e = \frac{1}{|H|}\underline{H}$$

$$\underline{H} = \sum_{h \in H} h \in R$$

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le D})$

G-orbits on $X \times X$

commutative association scheme

Assume $\ {}^tR_i = R_i$ all i (symmetric association scheme)

i.e.

$\forall x, y \in X \ \ \exists a \in G, \quad ax = y, \ ay = x$

generously transitive

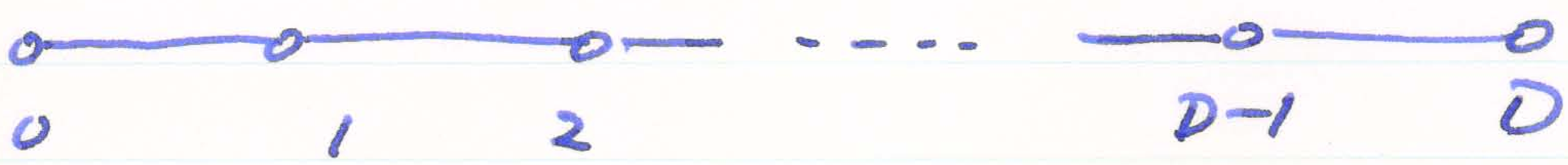
P-polynomial

$\Updownarrow \quad G = \bigcup_{i=0}^{D} H a_i H \qquad (a_0 = 1)$

$\underline{H a_1 H} \cdot \underline{H a_i H}$

$\in \mathbb{C}\, \underline{H a_{i-1} H} + \mathbb{C}\, \underline{H a_i H} + \mathbb{C}\, \underline{H a_{i+1} H}$, all i

distribution diagram w.r.t. $H a_1 H$

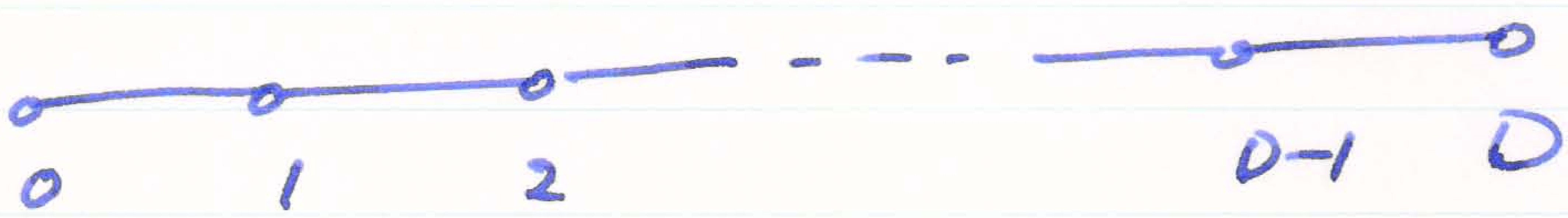


Q-polynomial

$\Uparrow \quad 1_H^G = \chi_0 + \chi_1 + \cdots + \chi_D$
$(\chi_0 = 1_G)$

representation diagram w.r.t. χ_1

$\chi_1 \chi_i \in \mathbb{C}\, \chi_{i-1} + \mathbb{C}\, \chi_i + \mathbb{C}\, \chi_{i+1}$



$\mathcal{X} = (X, \{R_i\}_{0 \le i \le D})$

(P and Q)-polynomial scheme

$\mathcal{A} = \langle A_0, A_1, \cdots, A_D \rangle$

$= \langle E_0, E_1, \cdots, E_D \rangle$

Bose-Mesner algebra

1st eigenmatrix $P = (P_j(i))$

$$A_j = \sum_{i=0}^{D} P_j(i) E_i$$

2nd eigenmatrix $Q = (Q_j(i))$

$$E_j = \frac{1}{|X|} \sum_{i=0}^{D} Q_j(i) A_i$$

$$P \cdot Q = Q \cdot P = n I$$

<u>parameters</u> of $\mathcal{X}$

$\{P_j(i) \mid 0 \le i \le D, \ 0 \le j \le D\}$

or equivalently

$\{Q_j(i) \mid 0 \le i \le D, \ 0 \le j \le D\}$

$$A_j = v_j(A), \qquad A = A_1$$

eigenvalues of A : $\theta_i = P_1(i)$, $0 \le i \le D$

eigenvalues of A_j : $v_j(\theta_i) = P_j(i)$, $0 \le i \le D$

$$nE_j = v_j^*(nE), \quad E = E_1, \quad n = |X|$$

$$\theta_i^* = Q_1(i), \qquad 0 \le i \le D$$

Then $Q_j(i) = v_j^*(i)$, $\quad 0 \le i \le D$, $\quad 0 \le j \le D$

$\{v_j(x)\}_{j=0}^{D}$ orthogonal polynomials

support $\{\theta_i\}_{i=0}^{D}$ weight $\{m_i\}_{i=0}^{D}$

$\{v_j^*(x)\}_{j=0}^{D}$ orthogonal polynomials

support $\{\theta_i^*\}_{i=0}^{D}$ $\{k_i\}_{i=0}^{D}$

$$m_i = \text{rank } E_i = Q_i(0)$$

$$k_i = \text{row sum of } A_i = P_i(0)$$

Then

$$\frac{v_j(\theta_i)}{k_j} = \frac{v_i^*(\theta_j^*)}{m_i}$$

Leonard Theorem

$$\left\{ \frac{v_i(x)}{k_i} \right\}_{i=0}^{D}, \quad \left\{ \frac{v_i^*(x)}{m_i} \right\}_{i=0}^{D}$$

are Askey-Wilson polynomials
or their limits.

(type I) parameters in terms of
$\Lambda_I = (r_1, r_2, s, s^*; h, h^*, \theta_0, \theta_0^* \mid q, D)$
$r_1 r_2 = s s^* q^{D+1}$

$$\theta_i = \theta_0 - h(1-q^{-i})(1-sq^{i+1}), \quad 0 \leq i \leq D$$

$$\theta_i^* = \theta_0^* - h^*(1-q^{-i})(1-s^*q^{i+1}), \quad 0 \leq i \leq D$$

$$\frac{v_i(x)}{k_i} = {}_4\phi_3\left(\begin{matrix} q^{-i}, & s^*q^{i+1}, & q^{-y}, & sq^{y+1} \\ q^{-D}, & r_1 q, & r_2 q & \end{matrix} ; q, q \right), \quad 0 \leq i \leq D$$

$$x = \theta_0 + h \frac{1}{q^y} (1-q^y)(1-sq^{y+1})$$

$$\frac{v_i^*(x)}{m_i} = {}_4\phi_3\left(\begin{matrix} q^{-i}, & sq^{i+1}, & q^{-y}, & s^*q^{y+1} \\ q^{-D}, & r_1 q, & r_2 q & \end{matrix} ; q, q \right), \quad 0 \leq i \leq D$$

$$x = \theta_0^* + \frac{h^*}{q^y} (1-q^y)(1-s^*q^{y+1})$$

(Type II) $q \to 1$

(type III) $q \to -1$ Bannai-Ito polynomials

Approach to the classification of
(P & Q)-polynomial schemes with large D

(A) show the parameters are those
in Bannai's list

<u>Observation</u>
core part : (1) $s^* = 0$

(2) type I : $q = p^f > 1$
for some prime p
except for $Her_D(r)$,
which has $q < -1$

(B) characterization by the parameters

present status

(A) (Bannai - Ito) (1) $\theta_i \in \mathbb{Z}$ all i

(2) $q = \pm 1$ or

$q \neq$ root of unity

(Terwilliger) $q \neq \pm 1$ can be assumed

(If $q = \pm 1$, then the classification is finished)

(B) $H(D, q)$ Egawa

$J(N, D)$ Terwilliger

$Her_D(r)$ Ivanov - Shpectrov

Terwilliger

$J_q(N, D)$ Metsch ($D \geq 3$)

except for $N = 2D, 2D+1$

$N = 2D+2$ + $q = 2, 3$

$N = 2D+3$ + $q = 3$

$Bil_{D \times N}(q)$ Metsch

except for $N = D, D+1, D+2$

$N = D+3$ + $q = 2$

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le D})$ symmetric association scheme

Terwilliger's theory

Terwilliger algebra w.r.t. $x_0 \in X$

$T = T(x_0) = \langle \mathcal{A}, \mathcal{A}^* \rangle \subseteq \mathrm{End}(V)$ ($V = \mathbb{C}X$ standard module)

$$\mathcal{A} = \langle \overset{I}{\overset{\|}{A_0}}, \overset{A}{\overset{\|}{A_1}}, \cdots, A_D \rangle$$

Bose-Mesner alg.

$$\mathcal{A}^* = \langle \underset{I}{\underset{\|}{A_0^*}}, \underset{A^*}{\underset{\|}{A_1^*}}, \cdots, A_D^* \rangle$$

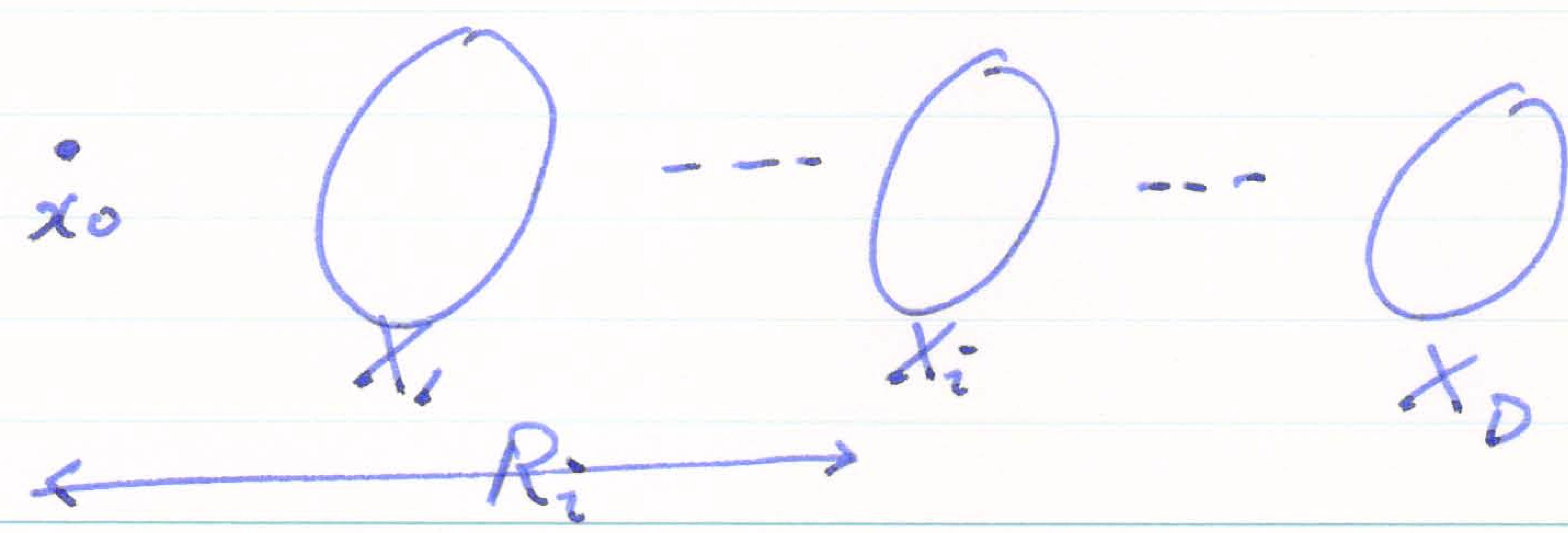
$$A_j^* \overset{\text{def}}{=} \sum_{i=0}^{D} Q_j(i) E_i^*$$

where

$$E_i^* : V \longrightarrow V_i^* = \mathbb{C}X_i$$

orthogonal projection

$$X_i = \{ x \in X \mid (x_0, x) \in R_i \}$$



$$I = E_0^* + E_1^* + \cdots + E_D^*$$

$$E_i^* E_j^* = \delta_{ij} E_i^*$$

$$J = A_0 + A_1 + \cdots + A_D$$

$$A_i \circ A_j = \delta_{ij} A_i$$ Hadamard product

$$A_j^* = \sum_{i=0}^{D} Q_j(i) E_i^*$$

$$\updownarrow$$

$$n E_j = \sum_{i=0}^{D} Q_j(i) A_i \qquad (n = |X|)$$

$$\mathcal{A}^* = \langle A_0^*, A_1^*, \cdots, A_D^* \rangle$$ dual Bose-Mesner algebra

w.r.t. ordinary matrix product

$$\simeq$$

$$\mathcal{A}^\circ = \langle nE_0, nE_1, \cdots, nE_D \rangle$$

w.r.t. Hadamard product

<u>Terwilliger algebra</u>

$$T = T(x_0) = \langle A, A^* \rangle$$ non-commutative semi-simple

$$V = \mathbb{C}X = \bigoplus_{i=0}^{D} V_i, \qquad V_i = E_i V$$

$$= \bigoplus_{i=0}^{D} V_i^*, \qquad V_i^* = E_i^* V$$

<u>P-polynomial</u> w.r.t. $A = A_1$

$$\Longleftrightarrow \quad A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad \text{all } i$$

<u>Q-polynomial</u> w.r.t. $E = E_1$, i.e., $A^* = A_1^*$

$$\Longleftrightarrow \quad A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad \text{all } i$$

$\mathcal{X} = (X, \{R_i\}_{0 \le i \le D})$

(P & Q)-poly. scheme

$X \ni x_0$ fixed

$T = T(x_0) = \langle \mathcal{A}, \mathcal{A}^* \rangle$

$= \langle A, A^* \rangle, \quad A = A_1, \; A^* = A_1^*$

$V \supset W$ irreducible T-module

$\|$

$\mathbb{C}X$ standard module

A, A^* are diagonalizable on W

$$W = \bigoplus_{i=0}^{d} W_i, \qquad W_i = E_{\mu+i} W$$

$$W = \bigoplus_{i=0}^{d} W_i^*, \qquad W_i^* = E_{\nu+i}^* W$$

eigenspace decompositions of A, A^*

$$A|_{W_i} = \theta_{i+\mu}, \quad A^*|_{W_i^*} = \theta_{i+\nu}$$

$$\begin{cases} A W_i^* \subseteq W_{i-1}^* + W_i^* + W_{i+1}^*, & \text{all } i \\ A^* W_i \subseteq W_{i-1} + W_i + W_{i+1}, & \text{all } i \end{cases}$$

Such a pair $A|_W, A^*|_W \in \mathrm{End}(W)$

is called a tridiagonal pair

(TD-pair)

TD-pair A, A^* on W is called <u>thin</u> if $\dim W_i = 1$, $\dim W_i^* = 1$ all i.

A thin TD-pair is called a <u>Leonard pair</u> (L-pair)

<u>principal T-module</u>

$W_0 = T x_0$. ($X \ni x_0$ the fixed base vertex)

- irreducible and thin
- contains the same information as the Bose-Mesner algebra, i.e., all the information of parameters

<u>Terwilliger</u>

Leonard Theorem

||

classification of L-pairs

$\cup$|

principal T-modules

$V \supset W$ irred. T-module

$$W = \bigoplus_{i=0}^{d} W_i^*, \qquad W_i^* = E_{\nu+i}^* W$$

$$\nu = \min\{ j \mid E_j^* W \neq 0 \}$$

<u>endpoint</u> of W

$$d = \#\{ j \mid E_j^* W \neq 0 \} - 1$$

<u>diameter</u> of W

(Ito-Terwilliger)

$$\dim W_0^* = 1 \qquad (\dim W_0 = 1)$$

<u>local eigenvalue</u> $a_0(W)$

the eigenvalue of $E_\nu^* A E_\nu^*$ on W_0

$$\mu = \min\{ j \mid E_j W \neq 0 \}$$

dual endpoint of W

$$W = \bigoplus_{i=0}^{d} W_i, \qquad W_i = E_{\mu+i} W$$

Suppose W is <u>thin</u>, i.e. $\dim W_i = 1$, $\dim W_i^* = 1$ all i

Then the isomorphism class of W is determined by

$$(\nu, \mu, d, a_0(W)).$$

<u>Terwilliger's lecture notes</u> (unpublished)

- If $\nu = 1$, then the isomorphism class of an irreducible T-module W is determined by the local eigenvalue $a_0(W)$

- # isomorphism classes of irreducible <u>thin</u> T-modules with $\nu = 1$ $\leqq 4$

the corresponding local eigenvalues $a_0(W)$ are the zeros of the <u>Terwilliger polynomial.</u>
(degree 4)

— If $\nu = 1$, a <u>non thin</u> irreducible T-module W has $d = D-1$
diameter

and
$$\dim W_0 = \dim W_d = 1$$
$$\dim W_i = 2, \quad 1 \le i \le d-1$$
and
$$\left(\begin{array}{l} \dim W_0^* = \dim W_d^* = 1 \\ \dim W_i^* = 2, \quad 1 \le i \le d-1 \end{array} \right)$$

Ito - Terwilliger Theory

type I $\quad V \supset W$ irred. T-module
$q \neq$ root of unity

$\exists$ algebra homomorphism

$$U_{\sqrt{q}}(\hat{sl}_2) \longrightarrow T\big|_W$$

quantum affine alg.

i.e. W has a $U_{\sqrt{q}}(\hat{sl}_2)$-module structure (tensor product of evaluation modules)

classification of TD-pairs (type I)

a TD-pair = a sort of tensor product of L-pairs

<u>Def</u> $\Gamma = (X, R)$ P-polynomial + Q-polynomial
(Distance Regular)

$X \ni x_0$ $T = T(x_0)$, Terwilliger alg

x_0' $T' = T(x_0')$

$V = \mathbb{C}X$ standard module

Γ is <u>pseudo distance-transitive</u>
(DT)

if ${}_{T}V \simeq {}_{T'}V$ (isomorphic)

for any $x_0, x_0' \in X$,

where T and T' are identified

by $E_i^* \longleftrightarrow E_i^{*\prime}$

$$\left(\begin{array}{l} T = \langle A, E_0^*, E_1^*, \cdots, E_D^* \rangle \\ T' = \langle A, E_0^{*\prime}, E_1^{*\prime}, \cdots, E_D^{*\prime} \rangle \end{array} \right)$$

<u>Rm</u> DT $\Longrightarrow$ pseudo DT.

Problem I Classify (P&Q)-polynomial schemes ~~[illegible]~~ with the properties (I), (II):

(I) pseudo DT.

(II) all irreducible T-modules are thin.

Problem II classify (P&Q)-polynomial schemes with the property (I).

Terwilliger might have expected that (I), (II) holded always.

counter examples:

twisted q-Johnson for (I)

affine schemes for (II)

(Gavrilyuk - Koolen)
characterization of $J_q(N, D)$, $N=2D$

zeros of the Terwilliger poly. = 3
one of the three zeros
is a double zero

they are actually the local eigenvalues
on the 1st subconstituent

irreducible T-modules with $\nu=1$
are thin

1st subconstituent $\cong$ q-clique
extension
of a grid.

The 1st + 2nd subconstituents
determine the global structure