Schur, Wielandt, Tamaschke: the development of S-rings as a tool for group theorists

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Outline

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Historical context

In the latter half of the 19th Century Berlin was possibly the leading place for mathematics in the world. But by the year 2000 the star of Göttingen was already shining brightly.
Frobenius (full professor in 1894) became the leader in Berlin.
Group representation theory began with Frobenius and Schur was Frobenius's star pupil. Schur's results include
(1)The representation theory of $GL(n, \mathbb{C})$.
(2) The exposition of group representation theory in what is essentially the modern way.
(3) The theory of projective representations of groups and the beginning of cohomology theory.
(4) The linear and projective representation theory of S_n and A_n.
At the same time Burnside was developing his own theorems in group theory, sometimes in parallel with Frobenius. The theory of S-rings developed from the alternative proof which Schur gave of one of Burnside's theorems.

Frobenius and S-rings

Another source of S-rings may be said to be the first paper of Frobenius on character theory of finite groups (1896). In it he uses the class algebra of the group to obtain the structure constants of the corresponding S-ring by solving equations and then produces the characters as eigenvalues. He further considers the fused class algebra coming from the embedding of a group G in a group H and as an example produces a "character table" for A_4 using the fused S-ring coming from the orbits of S_4 acting by conjugation. All this is before he connected characters with linear representations.

Schur and S-rings

The relevant papers of Schur for S-rings are the following:
(1) Neuer Beweis eines Satzes von W. Burnside, (A new proof of a theorem of Burnside),1908.
(2) Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendliche vielen Verändlichen, (Remarks on the theory of bounded bilinear forms with infinitely many variables) (1911)
(3) Zur Theorie der einfach transitiven Permutationsgruppen, (On the theory of simply transitive permutation groups), 1933.

The 1908 paper

In this Schur gives a new proof of the theorem of Burnside: Each permutation group on a set of p elements, where p is a prime, is either soluble or doubly transitive.
The techniques which he uses involve the form

$$F = \sum_{i,j=1}^{n} a_{i,j} x_i y_j,$$

and he lets a group G act on F by

$$g\left(\sum_{i,j=1}^{n} a_{i,j} x_i y_j\right) = \sum_{i,j=1}^{n} a_{i,j} x_{g(i)} y_{g(j)}.$$

The F which are invariant under elements of the group are analogous to the bases of an S-ring.

The 1911 paper

This paper is not so obviously connected to S-rings. He quotes papers of Hilbert and Hellinger on bilinear forms in an infinite number of variables and a paper by Hellinger and Toeplitz on "Foundations for a theory of infinite matrices". The paper sometimes uses forms in a finite number of variables and has been quoted as a source of a wide range of results and techniques in analysis. An algebraic result in the paper is quoted in a paper by Don Thompson: "The Krein condition for coherent configurations".

The 1933 paper

Here the centralizer ring of a permutation group G is defined as the ring of matrices which commute with a set of permutation matrices representing the elements of G. If G contains a regular subgroup H Schur explains how H may be identified with the set on which G acts. His work is explained more fully in Wielandt's book. However, as is often the case in Schur's papers he makes remarks which do not appear in later expositions. One of these is that certain matrices connected with basis elements of the centralizer ring are group matrices for H. The impact of this is that they are determined by their first row and therefore behave like sums of elements of H.

In (3) Schur mentions that the main theorem that a permutation group of degree n containing a cycle of order n cannot be simply primitive, which is actually an extension of the Burnside theorem which he reproved in the paper, has noteworthy applications to other problems of group theory and also to questions in number theory and analysis. He indicates that he will give these indications in a further paper. Circumstances intervened and no subsequent paper appeared. It seems that Schur was devastated by events in Germany never really recovered. Since Burnside's original proof by means of character theory has a gap in it, Schur's proof of the result is the first.

Wielandt

As a student of Schur he formalised the definition of S-ring and introduced the definition of B-group. His book on permutation groups had a wide influence. His lecture notes on infinite permutation groups and invariant relations also were influential. His work continues the tradition of Schur. It is immaculately presented and very thorough.
He comments on Schur's lecture style as follows: " He presented crystal clear finished theories polished to the last detail".

In Wielandt's acceptance of membership of the Heidelberg Akademie der Wissenschaften he makes the following comments. " To finish this report it is perhaps appropriate to say a few words to place my work into its general context. The main line of development of mathematics has been characterized for several decades by the invasion of the axiomatic method into ever more areas. The goal is to derive all of mathematics from just a few basic principles such as order and continuity... . It is as if some areas of mathematics which earlier could hardly be reached on foot are now connected by motorways. My own work has contributed nothing to these significant developments, except perhaps the recently undertaken attempt to free the theory of permutation groups from its restriction to finite groups. In fact, the impetus which Göttingen had given to abstract algebra reached Berlin just when I was a student, and recognition of the implication of the axiomatic method fascinated me just as it did my fellow students.

But I could not share the general opinion that this would henceforth be the only rewarding direction of research. It seemed to me that, like all great deductive systems, it was threatened by the danger that the problems which it could not properly accommodate would be dismissed as uninteresting, whereas on the contrary, these ought to provide a stimulus to broaden the foundations. ... .In terms of the metaphor I used earlier, this research area (of finite groups) seems to me to be a mountainous area that is still undisturbed by roads and has to be traversed on foot. But this has its charm. And the nice surprises that one experiences compensate for the occasional compassionate glances of motorists.

(The above translations are due to Molzon and Neumann)

Tamaschke

He was a student of Wielandt, and continued the work on S-ings and double coset algebras. He published two sets of lecture notes, one on permutation groups and the other on S-rings. He brought in categorical ideas and discussed "new" character theories for groups arising from S-rings. However, there do not appear to be many follow-up papers which apply his work.
The theory of association schemes/coherent configurations seems to have become the best framework to discuss the above aspects of S-rings, but the arguments of Schur and his followers often specifically need the presence of a regular subgroup.

B-groups

Definition

A B-group is a finite group H such that every primitive group containing the regular representation of H as a transitive subgroup is doubly transitive.

Schur's result in (3) may be stated as:
A cyclic group is a B-group.
Burnside originally conjectured that every abelian group is a B-group but this is false.

Definition

An S-ring $\mathbf{S} = \{\Gamma_i\}_{i=1}^k$ on G is *primitive* if no subgroup K except for $\{e\}$ or G is such that $\overline{K}$ lies in $\mathbf{S}$. If such a K exists then $\mathbf{S}$ is *imprimitive*.

Definition

The *trivial S-ring* on G is $\mathbf{T} = \{\{e\}, G - \{e\}\}$.

It follows that if every S-ring on G is either imprimitive or trivial then G is a B-group.

Examples of B-groups

(1) $\mathcal{C}_{p^a} \times \mathcal{C}_{p^b}$ where $a \neq b$.
(2) D_{2n}.
(3) The non-abelian group of order p^3 and exponent p^2 for $p > 3$.
The known B-groups are given in the article by P.M. Neumann in the collected works of Wielandt (Vol 1 PP 9, 13-15).

In particular a result on primitive groups by Cameron, Neumann and Teague implies that for almost all integers n every group of order n is a B-group. The proofs of some of the results are quite difficult, the result of Wielandt proving that D_{2n} is a B-group involving the theorem of Dirichlet on primes in an arithmetic progression.

The "rational" methods of Schur using S-ring techniques seem to have been more useful than the character theoretic methods first used by Burnside in investigating questions involving B-groups, and the papers which use arguments involving character theory have been unusually prone to error. A paper by Knapp has provided a proof of Burnside's result using character theory but he indicates that his completion of Burnside's proof is not very different in concept from the proof of Schur in that he is close to using a dual argument to that of Schur.

Non-B-groups

Neumann gives the following list of finite non-B-groups
(a) $X_1 \times X_2 \times \ldots \times X_k$ where each X_i has order the same order $m > 2$ and $k > 1$.
(b) Non-abelian simple groups.
(c) The non-abelian group of order p^3 and exponent p for $p > 3$.
(d) The non-abelian group of order 21.
(e) The non-abelian group of order 27 and exponent 9.
(f) The Frobenius group of order 992.

Infinite B-groups

The known results on infinite B-groups are all negative. For an element g of a group let the square root set of g be $\{x : x^2 = g\}$. If $g = e$ then call the corresponding square root set *principal* and otherwise define the square root set of g to be *non-principal*. It is shown in a paper Cameron-KWJ that if G is a group of countably infinite order which satisfies the condition that G cannot be represented as the union of a finite number of translates of a finite number of non-principal square root sets together with a finite set then G is not a finite B-group. In fact such a group G can be embedded as a regular subgroup of the automorphism group of the "universal" random graph, which is simply primitive.

In particular no countably infinite abelian group is a B-group. Peter Cameron has extended these ideas (St. Andrews talk).

Question: Can Schur's paper (2) on bilinear forms be used to prove results on infinite B-groups?

Takacs

In a set of talks from a mini-conference in 1983, published as Probability and Harmonic Analysis, there is a paper by Takacs "Harmonic analysis on Schur algebras and its applications in the theory of probability". In it he considers a function constant on the classes of an S-ring over a group G. He gives the Fourier Transform and Fourier inversion for such a function.Takacs seems to have had a distinguished career in probability theory but I have not found any references to other papers by him in the direction of S-rings. His paper refers to most of the literature then available on S-rings and association schemes. Among other things he discusses random walks (he actually calls them flights) on regular polytopes. However, in the applications which he discusses he does not seem to need an S-ring, but only an association scheme arising from the action of a group on a distance transitive graph. The lecture notes by Diaconis: Group Representations in Probability and Statistics set out similar work without any discussion of S-rings.