DEZA GRAPHS WITH PARAMETERS (n, k, k-1, a) AND $\beta = 1$

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Strongly regular graphs and Deza graphs

Definition. A non-empty k-regular graph Γ on n vertices is called a *strongly regular* with parameters (n, k, λ, μ), if the number of common neighbours of any two adjacent vertices is equal to λ and the number of common neighbours of any two distinct non-adjacent vertices is equal to μ.

Definition. A non-empty k-regular graph Γ on n vertices is called a *Deza graph with parameters* (n, k, b, a), where $n > k \geq b \geq a \geq 0$, if the number of common neighbours of any two distinct vertices takes the values a or b.

Definition. *A strictly Deza graph* is a Deza graph that is not a strongly regular graph and has diameter 2.

The concept of Deza graphs was introduced in the initial paper:

[EFHHH] M. Erickson, S. Fernando, W. H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs. *J. Comb. Designs.* **7** (1999) P. 359–405.

In this paper a basic theory of strictly Deza graphs was developed and several ways to construct such graphs were introduced. Moreover, all strictly Deza graphs with number of vertices at most 13 were found.

Later S. Goryainov and L. Shalaginov found all strictly Deza graphs whose number of vertices is equal to 14, 15, or 16. Also they found all strictly Deza graphs that are Cayley graphs with number of vertices less than 60.

[GSh1] S.V. Goryainov, L.V. Shalaginov, On Deza graphs with 14, 15 and 16 vertices, Siberian Electronic Math. Rep. 8 (2011) 105–115.

[GSh2] S.V. Goryainov, L.V. Shalaginov, Cayley–Deza graphs with less than 60 vertices, Siberian Electronic Math. Rep. 11 (2014) 268–310.

Strictly Deza graphs

Some problems arising in the theory of strictly Deza graphs are similar to those in the theory of strongly regular graphs. However, results and methods in these theories sometimes differ, and an analysis of these differences can enrich both theories.

For example, it is known that the vertex connectivity of a connected strongly regular graph equals its valency [BM]. The vertex connectivity of some class of strictly Deza graphs was investigated in [GGK]. It turns out, the only example of a strictly Deza graph exists, whose vertex connectivity is not equal to the valency.

[BM] A.E. Brouwer, D.M. Mesner, The connectivity of strongly regular graphs, Eur. J. Combin. 6 (3) (1985) 215–216.

[GGK] A.L. Gavrilyuk, S.V. Goryainov, V.V. Kabanov, On vertex connectivity of Deza graphs, Proc. Steklov Inst. Math. 285 (Suppl. 1) (2014) 68–77.

Strictly Deza graphs

Another example, recently G. Greaves and J. Koolen answered a question of A. Neumaier from 1981 about edge-regular graphs with regular clique. G. Greaves will probably talk about it.

[GK] G.R.W. Greaves, J.H. Koolen, Edge-regular graphs with regular cliques, European Journal of Combinatorics 71 (2018) 194–201.

Some other links to bibliography on Deza graphs could be found in the homepage of Michel Marie Deza
https://web.archive.org/web/20120404180819/http://www.liga.ens.fr/

Strong product of graphs

The *strong product* $\Gamma_1[\Gamma_2]$ is a graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$, and adjacency defined by as follows:

(u_1, u_2) is adjacent to (v_1, v_2) iff u_1 is adjacent to v_1 or $u_1 = v_1$ and u_2 is adjacent to v_2.

If a strongly regular graph Γ has parameters (n, k, λ, μ) with $k = \mu$, then Γ is a complete multipartite graph.

An analogues result for strictly Deza graphs with $b = k$ was obtained in [EFHHH].

Theorem 1. ([EFHHH], Theorem 2.6) *Let* Γ *be a Deza graph with parameters* (n, k, b, a). *Then* $k = b$ *if and only if* Γ *is isomorphic to the strong product* $\Gamma_1[\Gamma_2]$, *where* Γ_1 *is a strongly regular graph with paramters* (n_1, k_1, λ, μ), $\lambda = \mu$ *and* Γ_2 *is* $\overline{K_{n_2}}$ *for some* n_1, k_1, λ, n_2. *Moreover, the parameters satisfy* $n = n_1 n_2$, $k = k_1 n_2$, $a = \lambda n_2$, *and* $n_2 = \frac{k^2 - an}{k - a} \geq 2$.

The smallest example of such Deza graph we have from the triangular graph $T(6)$ with parameters $(15, 8, 4, 4)$ as Γ_1 and $\overline{K_2}$ as Γ_2.
It was D. Fon-Der-Flaass [DF] who has observed that construction of Wallis [WW] gives rise to more than exponentially many strongly regular graphs with various parameter sets, in particular with $\lambda = \mu$.

[DF] D. G. Fon-Der-Flaass, New prolific constructions of strongly regular graphs, Adv. Geom. 2 (2002), 301–306.

[WW] W. D. Wallis, Construction of strongly regular graphs using affine designs, Bull. Austral. Math. Soc. 4 (1971), 41–49.

Deza graphs with $b = k - 1$

If a strongly regular graph Γ has parameters (n, k, λ, μ) with $\lambda = k - 1$, then Γ is a disjoint union of cliques.

If a strongly regular graph Γ has parameters (n, k, λ, μ) with $\mu = k - 1$, then Γ is the pentagon.

The complement of a strongly regular graph Γ with parameters (n, k, λ, μ) is also strongly regular with parameters $(\overline{n}, \overline{k}, \overline{\lambda}, \overline{\mu}) = (n, n-k-1, n-2k+\mu-2, n-2k+\lambda)$. Therefore, if a strongly regular graph Γ has parameters (n, k, λ, μ), where $k = \mu$, then the parameters of its complement $\overline{\Gamma}$ satisfy the equality $\overline{k} = \overline{\lambda} + 1$. Hence, the structure of a strongly regular graph Γ with $k = \lambda + 1$ can be obtained from the corresponding result for a strongly regular graph with $k = \mu$ and vice versa.

The situation in the case of Deza graphs is quite different. Namely, let Γ be a Deza graph. Its complement $\overline{\Gamma}$ is a Deza graph only if and only if Γ has parameters $b = a + 2$ and is edge-regular or coedge-regular. Thus, there is no connection between Deza graphs with $k = b$ and Deza graphs with parameters satisfying $k = b + 1$.

Let Γ be a Deza graph with parameters (n, k, b, a). For a vertex x of Γ, put

$$A(x) := \{y \in V(\Gamma) : |N(x) \cap N(y)| = a\},$$

$$B(x) := \{y \in V(\Gamma) : |N(x) \cap N(y)| = b\},$$

Note that $|V(\Gamma)| = 1 + |A(x)| + |B(x)|$. Put

$$\beta(x) := |B(x)|.$$

It is known that $\beta(x)$ does not depend on the choice of a vertex. Moreover, $\beta(x)$ is uniquely determined by the parameters of Γ as follows:

$$\beta := \beta(x) = \frac{k(k-1)-a(n-1)}{b-a}.$$

The parameter β plays a key role in our investigation.

Theorem 2. ([KMSh]) *Let* Γ *be a strictly Deza graph with parameters* (n, k, b, a) *and* $\beta(\Gamma) > 1$. *The parameters* k *and* b *of* Γ *satisfy the condition* $k = b + 1$ *if and only if* Γ *is isomorphic to the strong product of* K_2 *with the complete multipartite graph with* $\frac{n}{n-k+1} > 1$ *parts of size* $\frac{n-k+1}{2}$.

Note that the adverb 'strictly' in Theorem 2 can not be removed, as is shown by the n-cycle with $n > 5$.

[KMSh] V.V. Kabanov, N.V. Maslova, L.V. Shalaginov,
On strictly Deza graphs with parameters $(n, k, k-1, a)$,
arXiv:1712.09529, December 2017.
Accepted to special issue of European Journal of Combinatorics dedicated to the memory of Michel Marie Deza.

Strictly Deza graphs with $b = k - 1$ and $\beta = 1$

Let Γ be a strictly Deza graph with parameters $(n, k, k-1, a)$ and $\beta = 1$. Then the vertices of Γ can be partitioned into pairs, where two vertices in every pair have b common neighbours, and two vertices from distinct pairs have a common neighbours.

This fact gives a connection with the notion of divisible design graphs.

Definition. A k-regular graph Γ on n vertices is a *divisible design graph* if the vertex set can be partitioned into t classes of size m, such that two distinct vertices from the same class have exactly λ_1 common neighbours, and two vertices from different classes have exactly λ_2 common neighbours.

[HKM] W.H. Haemers, H. Kharaghani, M.A. Meulenberg, Divisible design graphs, Journal of Combinatorial Theory, Series A. 118 (2011), 978–992.

Two constructions of Deza graphs with $\beta = 1$

We present two constructions of strictly Deza graphs with parameters $(n, k, k-1, a)$ and $\beta = 1$.

Both constructions use a strongly regular graph Δ with parameters (m, ℓ, λ, μ) where $\lambda = \mu - 1$.

First construction of Deza graphs with $\beta = 1$

Let Δ be a strongly regular graph with parameters (m, ℓ, λ, μ) where $\lambda = \mu - 1$.

Construction 1. *Let Γ be the strong product of Δ with K_2. The graph Γ is a Deza graph with parameters $(n, k, k-1, a)$ and $\beta = 1$, where $n = 2\ell$, $k = 2\ell + 1$, $a = 2\mu$.*

Examples for Δ:

- conference graphs
- the switched conference graphs
- Petersen graph
- Hoffman-Singleton graph

Note that Δ has $\lambda = \mu - 1$ iff the complementary graph $\overline{\Delta}$ has the same property.

Dual Seidel switching

Let B and A be the adjacency matrices of Δ and Γ, respectively, in Construction 1. Then $A = B \otimes J_2 - I_n$ (J_2 is the 2×2 all-ones matrix, and I_n is the identity matrix of order n).

Suppose that Δ has an involution φ that interchanges only non-adjacent vertices.

Let P be the corresponding permutation matrix. Then $B' = PB$ is a symmetric matrix (because $P = P^\top$ and $PBP = B$) with zero diagonal (because P interchanges only nonadjacent vertices).

So B' is the adjacency matrix of a graph Δ' (say), which is a Deza graph because $B'^2 = PBPB = B^2$.

This construction was given in [H] and the method has been called *dual Seidel switching*; see [H].

[H] W.H. Haemers, Dual Seidel switching, EUT Report 84-WSK-03, Eindhoven University of Technology, The Netherlands, 1984, pp. 183–190.

Point of view on dual Seidel switching

The following Lemma shows what is the neighbourhood of a vertex of the graph Δ'.

Lemma 1. *For the neighbourhood $\Delta'(x)$ of a vertex x of the graph Δ', the following condition holds:*

$$\Delta'(x) = \begin{cases} \Delta(x), & \textit{if } \varphi(x) = x; \\ \Delta(\varphi(x)), & \textit{if } \varphi(x) \neq x. \end{cases}$$

In other words, the dual Seidel switching precisely swaps the neighbourhoods $\Delta(x)$ and $\Delta(y)$, for all moved vertices x, y, $y = \varphi(x)$.

Second construction of Deza graphs with $\beta = 1$

Let Γ be the strong product of K_2 with Δ.

For any transposition $(x\ y)$ of the involution φ, modify Γ as follows:

- take the corresponding two pairs of vertices x', x'' and y', y'' in Γ
- delete the edges $\{x', x''\}$ and $\{y', y''\}$
- insert the edges $\{x', y''\}$ and $\{x'', y'\}$

Define Γ' to be the resulting graph. If A' is the adjacency matrix of Γ', then we can also construct Γ' from Γ with using dual Seidel switching as

$$A' = P_1 A, \text{ where } P_1 = P \otimes I_2.$$

We easily have that $(A')^2 = A^2$, which shows that Γ' is a Deza graph with the same parameters as Γ.

Second construction of Deza graphs with $\beta = 1$

Construction 2. *The graph* Γ' *is a Deza graph with parameters* $(n, k, k-1, a)$ *and* $\beta = 1$*, where* $n = 2m$*,* $k = 2\ell + 1$*,* $a = 2\mu$*.*

Note that in Γ any two vertices with b common neighbours are adjacent. For Γ' it is not true, therefore Γ and Γ' are non-isomorphic.

Our result

Theorem (S.V. Goryainov, W.H. Haemers, V.V. K., L.V. Shalaginov)
Let Γ be a Deza graph with parameters $(n, k, k-1, a)$, $k > 1$ and $\beta = 1$. Then Γ can be obtained from Construction 1 or Construction 2.

In case $k = 1$, Γ consists of $n/2$ disjoint edges and $\beta = 1$ implies $\Gamma = K_2$.

[GHKSh] S.V. Goryainov, W.H. Haemers, V.V. Kabanov, L.V. Shalaginov, Deza graphs with parameters $(n, k, k-1, a)$ and $\beta = 1$, arXiv:1806.03462, June 2018.

Let q be a power of odd prime p and $q \equiv 1 \pmod 4$.

Let $\mathbb{F}_q$ be the field of order q and ω be a primitive root of $\mathbb{F}_q$.

Denote by S_1 the set of all even powers of ω in $\mathbb{F}_q$.

Definition. *The Paley graph* $P(q)$ is the graph on $\mathbb{F}_q$ with two vertices being adjacent iff their difference belongs to S_1.

Denote by φ the automorphism of $P(q^2)$ that sends γ to γ^q. Note that φ fixes the elements from $\mathbb{F}_q$.

Lemma 2. *For any $\gamma = x + y\alpha$ from $\mathbb{F}_{q^2}$, the following holds:*
(1) $\gamma^q = x - y\alpha$;
(2) $\gamma - \gamma^q = 2y\alpha$.

Since α is a square iff $q \equiv 3(4)$, we have.

Lemma 3.
(1) If $q \equiv 1(4)$, then φ interchanges only non-adjacent vertices.
(2) If $q \equiv 3(4)$, then φ interchanges only adjacent vertices.

Thus, for any q, either the Paley graph $P(q^2)$ or its complement has an involution satisfying the condition of Construction 2.

Let Δ be a Paley graph $P(q^2)$ with the parameters $(4\mu+1, 2\mu, \mu-1, \mu)$, where $\mu = \frac{q^2-1}{4}$.

Take an order 2 automorphism of $P(q^2)$ that interchanges only nonadjacent vertices.

According to Construction 2, we obtain a new strictly Deza graph with parameters $(8\mu+2, 4\mu+1, 4\mu, 2\mu)$.

The Hoffman-Singleton graph, which is strongly regular with parameters $(50, 7, 0, 1)$, has a unique involutive automorphism ϕ that interchanges only non-adjacent vertices.

The Deza graph obtained from Hoffman-Singleton graph with using dual Seidel switching, has diameter 3.

However, each of Constructions 1 and 2 produces a strictly Deza graph with parameters $(100, 15, 14, 2)$.

Construction 1 applied to the complement gives a strictly Deza graph with parameters $(100, 85, 84, 72)$.

Symmetric conference matrices

An $m \times m$ matrix C with zero's on the diagonal, and ± 1 elsewhere, is a *conference matrix* if $CC^\top = (m-1)I$. If a conference matric C is symmetric with constant row (and column) sum r, then $r = \pm\sqrt{m-1}$, and $B = \frac{1}{2}(J_m - I_m - C)$ is the adjacency matrix of a strongly regular graph with parameter set

$$\mathcal{P}(r) = (\ r^2+1,\ \tfrac{1}{2}(r^2-r),\ \tfrac{1}{4}(r-1)^2-1,\ \tfrac{1}{4}(r-1)^2\ ).$$

Note that $\mathcal{P}(-r)$ is the complementary parameter set of $\mathcal{P}(r)$.

Symmetric conference matrices with constant row sum were constructed by Seidel.

If q is an odd prime power and $r = \pm q$, then such a conference matrix can be obtained from the Paley graph of order q^2. Let B' be the adjacency matrix of $P(q^2)$, and put $S = J_{q^2} - I_{q^2} - 2B'$ (S is the so-called *Seidel matrix* of $P(q^2)$).

Define

$$C' = \begin{bmatrix} 0 & \mathbf{1}^\top \\ \mathbf{1} & S \end{bmatrix}$$

($\mathbf{1}$ is the all-ones vector). Then C' is a symmetric conference matrix of order $m = q^2 + 1$. However, C' doesn't have constant row sum.

Next we shall make the row and column sum constant by multiplying some rows and the corresponding columns of C' by -1.

This operation is called *Seidel switching*, and it is easily seen that Seidel switching doesn't change the conference matrix property.

To describe the required rows and columns, we use the notation and description of $P(q^2)$ given in the previous slides.

If $q \equiv 3 \mod 4$ we take the complement of the described Paley graph. Then the involution φ given in above slide interchanges only non-adjacent vertices in all cases.

For $x \in \mathbb{F}_q$ define $V_x = \{x + y\alpha \mid y \in \mathbb{F}_q\}$.

Then the sets V_x form a partition of the vertex set of $P(q^2)$, and each class is a coclique. Moreover, the partition is fixed by the involution φ.

Let V be the union of $\frac{1}{2}(q-1)$ classes V_x. Then V induces a regular subgraph of $P(q^2)$ of degree $\frac{1}{4}(q-1)^2-1$ with $\frac{1}{2}q(q-1)$ vertices.

Now make the matrix C by Seidel switching in C' with respect to the rows and columns that correspond with V. Then C is a regular symmetric conference matrix, and $B=\frac{1}{2}(J-I-C)$ is the adjacency matrix of a strongly regular graph Γ with parameter set $\mathcal{P}(q)$, and φ remains an involution that interchanges only nonadjacent vertices.

Thus, Γ satisfies the conditions of Constructions 1 and 2.

We obtain strictly Deza graphs with parameters

$$\left(q^2+1,\ \frac{1}{2}(q^2-q),\ \frac{1}{4}(q-1)^2,\ \frac{1}{4}(q-1)^2-1\ \right)$$

(by dual Seidel switching),

$$\left(2q^2+2,\ q^2-q+1,\ \frac{1}{2}(q^2-q),\ \frac{1}{2}(q-1)^2\ \right)$$

(by Construction 1 and 2),
and

$$\left(2q^2+2,\ q^2+q+1,\ \frac{1}{2}(q^2+q),\ \frac{1}{2}(q+1)^2\ \right)$$

(by Construction 1 applied to the complement).

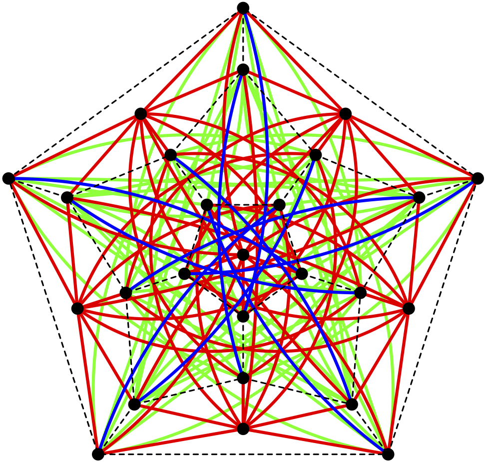
If $q=3$, Γ is the Petersen graph. It has one conjugacy class of involutive automorphisms that interchanges only non-adjacent vertices. The Deza graph obtained from the Petersen graph with dual Seidel switching has diameter 3. However, for $q>3$, the obtained Deza graphs are strictly Deza.

The Paulus-Rozenfeld-Thompson graph T was independently discovered at least three times at Eindhoven (1973), Moscow (1973) and Tucson (1979). It is one of the ten strongly regular graphs with the parameters $(v, k, \lambda, \mu) = (26, 10, 3, 4)$.

Among these 10 graphs the SRG T has the largest group $G = Aut(T)$ of order 120, which is isomorphic to $A_5 \times Z_2$, the full symmetry group of the dodecahedron.

There exists one conjugacy class of involutions which interchanges only nonadjacent vertices. Hence, we have one strictly Deza graph from Construction 1 and one strictly Deza graph from Construction 2 with parameters $(52, 21, 10, 8)$. Also there exists one involution which interchanges only adjacent vertices. Hence, we have one strictly Deza graph from Construction 1 and one strictly Deza graph from Construction 2 with parameters $(52, 31, 18, 16)$.

Logo of the conference



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