On the automorphism groups of primitive coherent configurations

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Plan for this talk

- Babai's **classification** of strongly regular graphs (**SRG**) by a measure of symmetry.
- Generalization to **distance-regular graphs** (**DRG**)
- Generalization to **coherent configurations (CC)**. Conjectures and known results.

Goal

Goal: Classify all PCC $\mathfrak{X}$ with “large" $G = \mathrm{Aut}(\mathfrak{X})$.

There are different ways to measure “largeness":

Most desired: **large order** of the group $|G|$.
$(|G| > \exp((\log n)^c))$.

We focus on:

large thickness of a group $\theta(G)$ $(\theta(G) > (\log n)^c)$

small minimal degree $mindeg(G)$ $(mindeg(G) < \varepsilon n)$.

Minimal degree of a permutation group

Degree $\deg(\sigma)$ **of permutation** $\sigma \in Sym(\Omega)$
is the number of points in Ω **not fixed** by σ.

Definition

Minimal degree of G is

$$\mathrm{mindeg}(G) = \min_{\sigma \neq 1}[\deg(\sigma)]$$

Examples: $\mathrm{mindeg}(S_n) = 2$, $\mathrm{mindeg}(A_n) = 3$,
for $\mathbb{Z}_n \leq S_n$: $\mathrm{mindeg}(\mathbb{Z}_n) = n$.

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Theorem (Bochert 1892)

If G is doubly transitive, $G \leq S_n$, then

$$\mathrm{mindeg}(G) \geq n/4$$

with known exceptions.

Only exceptions are: A_n, S_n.

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Theorem (Liebeck 1984, using CFSG)

If $G \leq S_n$ is a primitive permutation group, then

$$\mathrm{mindeg}(G) \geq \frac{n}{9\log_2(n)}$$

with known exceptions.

Exceptions are Cameron groups.

Structural consequences of large minimal degree

A group H is **involved** in G if $H \cong L/K$ for $K \triangleleft L \leq G$.

Terminology (Babai)

The maximal t for which A_t **is involved** in G is the **thickness** $\theta(G)$ of G.

Large minimal degree $\Rightarrow$ small thickness

Lemma (Wielandt 1934)

$G \leq S_n$ and minimal degree satisfies $\mathrm{mindeg}(G) \geq \Omega(n)$, then

$$\theta(G) \leq O(\log(n)).$$

Classification of SRG by measure of symmetry

Theorem (Babai 2014)

X strongly regular graph

$$\Rightarrow \quad \mathrm{mindeg}(\mathrm{Aut}(X)) \geq n/8$$

with known exceptions.

Our paper generalizes this to:

- Distance-regular graphs.
- Primitive coherent configurations of rank 4.

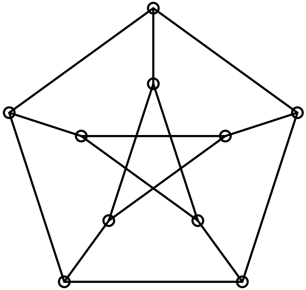
Techniques:

- structural combinatorics
- spectral graph theory

Notation: Strongly regular graphs

Strongly regular graph (**SRG**) with parameters (n, k, λ, μ)

n - number of vertices;
k - degree of every vertex;
$\lambda = |N(u) \cap N(v)|$ for $u \sim v$;
$\mu = |N(u) \cap N(v)|$ for $u \nsim v$.



$(n, k, \lambda, \mu) = (10, 3, 0, 1)$

Coherent configurations

V is a finite set. $\mathcal{R} = \{R_1, R_2, ..., R_s\}$ is a partition of $V \times V$.

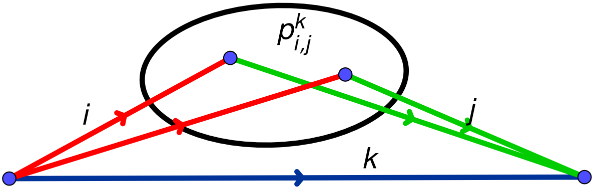
Partition $\mathcal{R} = (R_1, R_2, ..., R_s) \quad \longleftrightarrow \quad$ coloring $c : V \times V \to [s]$

$c(w, w)$ – a **vertex color** $\quad c(u, v)$ – an **edge color** if $u \neq v$

$\mathfrak{X} = (V, \mathcal{R}) = (V, c)$ is a **coherent configuration (CC)** if

1. vertex colors $\neq$ edge colors
2. $c(u, v)$ determines $c(v, u)$
3. $p^k_{i,j} = |\{w : c(u, w) = i, c(w, v) = j\}|$, where $c(u, v) = k$

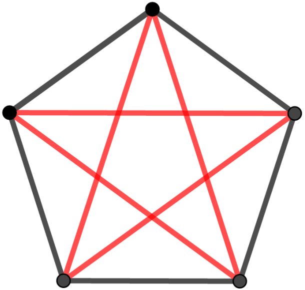
Def: s is a **rank** of CC.

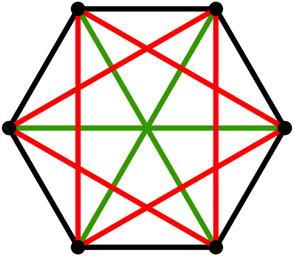


Special classes of coherent configurations

Special classes of CCs:

- **Homogeneous** CC.: $c(u,u) = c(v,v)$ for all $u, v \in V$.
- **Association schemes**: $c(u,v) = c(v,u)$ for all $u, v \in V$ $\Rightarrow$ homogeneous.
- **Metric** schemes: colors = distances in **DRG**,
  $$c(x,y) = \mathrm{dist}(x,y).$$
- **Primitive** coherent configurations (PCC): homogeneous + every constituent digraph is strongly (= weakly) connected

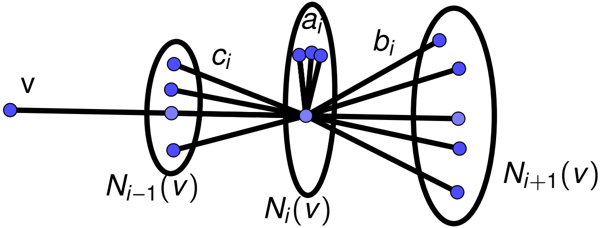




Distance-regular graphs

X is a **distance-regular graph** if $\exists$ sequence of parameters

$$\iota(X) = \{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots c_d\}, \text{ s.t.}$$



Notation: $b_0 = k$, $a_1 = \lambda$, $c_2 = \mu$. $\quad k_i = |N_i|$.

We have the following properties

$$a_i + b_i + c_i = k,$$

$$b_0 \geq b_1 \geq \cdots \geq b_{d-1} \quad \text{and} \quad c_1 \leq c_2 \leq \cdots \leq c_d.$$

Examples: Johnson graph

Johnson graph $J(m,t)$, $m \geq 2t+1$:

- $V = V(J(m,t)) = \binom{[m]}{t}$,
- $A, B \in V$ are adjacent iff $|A \setminus B| = |B \setminus A| = 1$.

$J(m,t)$ is DRG of diameter t. Smallest eigenvalue $-t$.

$$\mathrm{Aut}(J(m,t)) = S_m^{(t)} \quad \Rightarrow \mathrm{mindeg}(\mathrm{Aut}(J(m,t))) \leq O(n^{1-1/t})$$

Examples: Hamming graph

Hamming graph $H(t,m)$:

- $V = V(H(t,m)) = [m]^t$, i.e., strings of length t over $[m]$.
- $A, B \in V$ are adjacent if Hamming distance $d_H(A,B) = 1$.

$H(s,m)$ is DRG of diameter t. Smallest eigenvalue $-t$.

$$\mathrm{Aut}(H(t,m)) = S_m \wr S_t \quad \Rightarrow \mathrm{mindeg}(\mathrm{Aut}(H(t,m))) \leq O(n^{1-1/t})$$

Our result for DRG

Theorem (K. 2018)

*$\forall d \geq 3$, s.t. for any **primitive DRG** X of diameter d one of the following is true.*

1. $\mathrm{mindeg}(\mathrm{Aut}(X)) \geq \Omega(n)$.
2. *X is a **geometric** DRG.*

In the case of **diameter 3** we get **complete classification** of all exceptions to (1) even without primitivity assumption.
They are:

1. Johnson graph $J(s, 3)$,
2. Hamming graph $H(3, s)$,
3. cocktail-party graph.

A classification of PCC of rank 4

Theorem (K. 2018)

Let $\mathfrak{X}$ be a PCC of rank 4 on n vertices . Then one of the following is true

1. $\mathrm{mindeg}(\mathrm{Aut}(\mathfrak{X})) \geq \Omega(n)$
2. $\mathfrak{X}$ *is a Hamming or a Johnson scheme.*

Ideas of the proofs

Classification of SRG by measure of symmetry

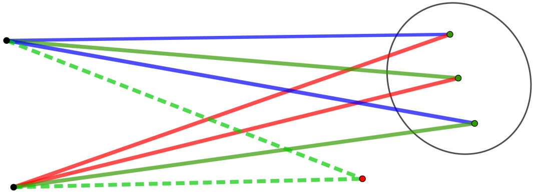
Theorem (Babai, 2014)

If X is a strongly regular graph on $n \geq 29$ vertices. Then one of the following is true

1. *minimal degree of* $\mathrm{Aut}(X)$ *is* $\geq n/8$.
2. *X or its complement is*
   - *triangular graph* $T(s) = J(s, 2)$,
   - *lattice graph* $L_2(s) = H(2, s)$,
   - *union of cliques (trivial case).*

Distinguishing number

A vertex x **distinguishes** u and v if $c(x,u) \neq c(x,v)$.



$D(u,v)$ = number of vertices that distinguish u and v.

Distinguishing number (Babai 1981):

$$D(X) = \min_{u \neq v \in V} D(u,v)$$

$$D(X) \leq \mathrm{mindeg}(\mathrm{Aut}(X)).$$

Note, if X is SRG, then $D(X) = 2(k - \max(\lambda, \mu))$.

Spectrum of DRG

Spectrum of X = eigenvalues of adjacency matrix.

Let X be **DRG of diameter d**, then its eigenvalues are the eigenvalues of $(d+1)\times(d+1)$ matrix

$$T(X) = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \dots \\ c_1 & a_1 & b_1 & 0 & \dots \\ 0 & c_2 & a_2 & b_2 & \dots \\ \dots & & \vdots & & \dots \\ \dots & & 0 & c_d & a_d \end{pmatrix}$$

Thus, there are $\boldsymbol{d+1}$ distinct **eigenvalues.**

For **SRG** eigenvalues are k and the solutions $\theta_2 \le \theta_1$ to

$$\theta_1 + \theta_2 = \lambda - \mu, \quad \theta_1\theta_2 = \mu - k.$$

Spectral tool

X is k-regular graph, $k = \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ is its spectrum.

Definition: **zero-weigth spectral radius**

$$\xi = \max\{|\theta_2|, |\theta_3|, \ldots, |\theta_n|\}$$

Note: $k - \xi$ is a **spectral gap** $\sim$ related to **expansion** of graphs.
$q = \max \#$ of common neighbors $\forall u \neq v \in V(X)$.

Lemma (Babai 2014)

$$\mathrm{mindeg}(\mathrm{Aut}(X)) \geq n\left(1 - \frac{\boldsymbol{q + \xi}}{\boldsymbol{k}}\right)$$

Lemma is proved using Expander Mixing Lemma.

Two tools

k - degree of graph X
ξ - zero-weight spectral radius
$D(X)$ - distinguishing number

Distinguishing number tool

$$\mathrm{mindeg}(\mathrm{Aut}(X)) \geq D(X)$$

Spectral tool

$$\mathrm{mindeg}(\mathrm{Aut}(X)) \geq n\left(1 - \frac{q+\xi}{k}\right)$$

Skeleton of the Babai's proof

Theorem (Seidel 1968)

If X is a non-trivial SRG on $n \geq 29$ vertices and least eigenvalue -2, then X is $T(s)$ or $L_2(s)$ for some s.

Skeleton of the Babai's proof

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If X is a non-trivial SRG on $n \geq 29$ vertices and least eigenvalue -2, then X is $T(s)$ or $L_2(s)$ for some s.

Note: complement of SRG is SRG, so we may assume $k \leq \frac{n-1}{2}$.
Then $\max(\lambda, \mu) \leq \frac{3}{4}k$. ($\Delta$ ineq. for distinguishing numbers)

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Case 1: $k \geq n/4 \Rightarrow D(X) = 2(k - \max(\lambda, \mu)) \geq n/8$
Apply **distinguishing number** tool.

Case 2: $k \leq \frac{n}{4}$ and smallest eigenvalue of X is at most -3.
Spectral tool works!

Case 3: Smallest eigenvalue is > -3.
These graphs are known by **Seidel's classification.**

The same framework for DRG

Step 1: Apply **distinguishing number** tool whenever possible.

Step 2: In remaining cases apply **spectral analysis**.

Step 3: Reduce to **classification results** for graphs with special properties in terms of $k, \lambda, \mu, \theta_{min}$.

Strategy for DRG

Let X be DRG of fixed diameter d.

1. If distinguishing number tool works. ✓
2. Else:
   - $\mu < \varepsilon k$
   - a_i approximate eigenvalues well

     $+$ **inequalities involving parameters b_i and c_i**

     $\Rightarrow$ X is an **expander** with spectral gap $> \gamma k$
     ($0 < \gamma < 1$ constant dependent on d)

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   1. If $\lambda \leq \gamma k/2$, then **spectral tool** works.
      Recall: $\mathrm{mindeg}(X) \geq n\left(1 - \frac{q+\xi}{k}\right)$

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   1. If $\lambda \leq \gamma k/2$, then **spectral tool** works.
      Recall: $\mathrm{mindeg}(X) \geq n\left(1 - \frac{q+\xi}{k}\right)$
   2. If $\lambda > \gamma k/2$, then X is **geometric.**

Geometric DRG

Theorem (Delsarte 1973)

X - DRG, θ_{min} - smallest eigenvalue, C - clique in X. Then

$$|C| \leq 1 - \frac{k}{\theta_{min}}$$

Definition

A clique of size $1 - \frac{k}{\theta_{min}}$ is called a **Delsarte clique.**

Definition

- Graph X contains **clique geometry**, if $\exists\, \mathscr{C}$ a collection of max. cliques s.t. $\forall$ **edge is in precisely 1 clique** $C \in \mathscr{C}$**.**
- If all $C \in \mathscr{C}$ are **Delsarte clique**, DRG X is **geometric.**

Corollaries from Metsch's criteria

Theorem (Metsch 1995)

Let X be a graph. Some system of inequalities on parameters of X implies that it has ***clique geometry****.*

Lemma (Spielman, Babai-Wilmes, Cor. to Metsch criteria)

If X satisfies $k\mu = o(\lambda^2)$, then X has clique geometry.

Lemma (Corollary to Metsch criteria)

Let X be DRG and $m \in \mathbb{N}$ with

- $(m-1)(\lambda+1) < k \leq m(\lambda+1)$,
- $\lambda \geq \frac{1}{2}m(m+1)\mu$.

Then X is ***geometric*** *with smallest eigenvalue $-m$.*

Our result for DRG

Theorem (K 2018)

$\forall d \geq 3\ \exists \gamma_d > 0, m_d$, s.t. for any primitive DRG X of diameter d one of the following is true:

1. $\mathrm{mindeg}(\mathrm{Aut}(X)) \geq \gamma_d n$.
2. *X is **geometric** with **smallest eigenvalue** $-m$, where $m \leq m_d$.*

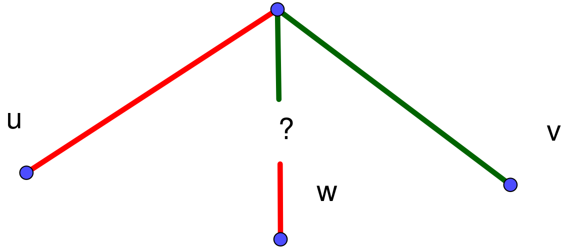
Furthermore, it is possible to take

$$m_d = \lfloor 2(d-1)(d-2)^{\log_2(d-2)} \rfloor.$$

More on Distinguishing numbers

Lemma (Properties of distinguishing number)

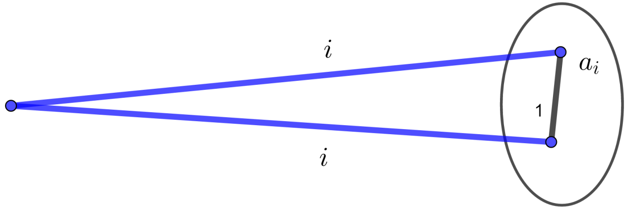
1. *$D(u,v)$ depends only on the color $c(u,v) = i$, so we can define $D(i) := D(u,v)$.*
2. *$D(u,v) \leq D(u,w) + D(w,v)$.*
3. *$D(X) \geq \frac{D(i)}{\mathrm{diam}(i)}$, where $\mathrm{diam}(i)$ is an undirected diameter color-i constituent.*



Distinguishing number for DRG

Recall, $D(X) \geq D(1)/d$. Recall, $k_i = |N_i(v)|$. We have,

$$D(1) \geq 2(k_i - p^1_{i,i}) = 2k_i \frac{(k - a_i)}{k}.$$



$k_i \geq \frac{n-1}{d}$ for some i $\implies$ **We want:** $\boldsymbol{k - a_i \geq \Omega(k)}$ for all i.

Bad case: $k - a_i$ is small for some i.
But: $k = a_i + b_i + c_i$, so b_i and c_i are small.
$\implies$ **we can do spectral analysis.**

Distinguishing number for DRG

Case 1: $k - a_i \geq \Omega(k)$ for all i.

Lemma

Let X be DRG of diameter d. Suppose for some $1 \leq i \leq d$

$$b_{i-1} \geq \alpha k, \quad c_i \geq \beta k.$$

Then

$$D(X) \geq \frac{\min(\alpha, \beta)}{d^2} n$$

Approximation of eigenvalues

Matrices are entry-wise close $\Rightarrow$ their eigenvalues are close

Theorem (Ostrowski 1960)

Let $A, B \in M_n(\mathbb{C})$.
$\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A
$\mu_1, \mu_2, \ldots, \mu_n$ are eigenvalues of B.

$$M = \max\{|(A)_{ij}|, |(B)_{ij}| : 1 \leq i, j \leq n\}, \quad \delta = \frac{1}{nM} \sum_{i=1}^{n} \sum_{j=1}^{n} |(A)_{ij} - (B)_{ij}|.$$

Then, there exists a permutation $\sigma \in S_n$ such that

$$|\lambda_i - \mu_{\sigma(i)}| \leq 2(n+1)^2 M \delta^{1/n}, \quad \textit{for all } 1 \leq i \leq n.$$

Spectral analysis of DRG

The eigenvalues of X are the eigenvalues of $T(X)$

$$T(X) = \left(\begin{array}{cc|c|cccc}
a_0 & b_0 & 0 & 0 & 0 & 0 & \dots \\
c_1 & a_1 & b_1 & 0 & 0 & 0 & \dots \\
\hline
0 & c_2 & \boldsymbol{a_2} & b_2 & 0 & 0 & \dots \\
\hline
0 & 0 & c_3 & a_3 & b_3 & & \\
\dots & \dots & & & \vdots & & \dots \\
\dots & 0 & 0 & 0 & c_{d-1} & a_{d-1} & b_{d-1} \\
\dots & 0 & 0 & 0 & 0 & c_d & a_d
\end{array}\right)$$

Suppose that $b_i < \varepsilon k$ and $c_i < \varepsilon k$ for some i.

Spectral analysis of DRG

The eigenvalues of X are the eigenvalues of $T(X)$

$$T(X) \approx \left(\begin{array}{cc|c|cccc} a_0 & b_0 & 0 & 0 & 0 & 0 & \dots \\ \color{red}{0} & a_1 & b_1 & 0 & 0 & 0 & \dots \\ \hline 0 & \color{red}{0} & \boldsymbol{a_2} & \color{red}{0} & 0 & 0 & \dots \\ \hline 0 & 0 & c_3 & a_3 & \color{red}{0} & & \\ \dots & \dots & & & \vdots & & \dots \\ \dots & 0 & 0 & 0 & c_{d-1} & a_{d-1} & \color{red}{0} \\ \dots & 0 & 0 & 0 & 0 & c_d & a_d \end{array}\right)$$

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Suppose that $b_i < \varepsilon k$ and $c_i < \varepsilon k$ for some i.

Eigenvalues are close to the a_j

$$|\theta_j - a_{\tau(j)}| \leq 2(d+2)^2 \epsilon^{1/(d+1)} k, \quad \text{for all } 1 \leq j \leq n.$$

Spectral analysis of DRG

The eigenvalues of X are the eigenvalues of $T(X)$

$$T(X) = \left(\begin{array}{cc|c|cccc}
a_0 & b_0 & 0 & 0 & 0 & 0 & \dots \\
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\dots & \dots & & & \vdots & & \dots \\
\dots & 0 & 0 & 0 & c_{d-1} & a_{d-1} & b_{d-1} \\
\dots & 0 & 0 & 0 & 0 & c_d & a_d
\end{array}\right)$$

Suppose that $b_i < \varepsilon k$ and $c_i < \varepsilon k$ for some i.

Eigenvalues are close to the a_j

We know $a_i \approx k$. So for large **spectral gap**:
Need: $(\forall j \neq i)\ \ k - a_j \geq \Omega(k) \Rightarrow$ **Need:** $(\forall j \neq i)\ \ b_j$ or $c_j \geq \Omega(k)$

Control of the expansion

Idea: If b_j is large and c_{j+1} is small, then b_{j+1} or c_{j+2} is large.

Lemma (K. 2018)

Let X be a primitive DRG, $d \geq 3$. Let $1 \leq j \leq d-2$ and $\alpha_j > \varepsilon > 0$. Suppose that $c_{j+1} \leq \varepsilon k$ and $b_j \geq \alpha_j k$. Denote $C = \frac{\alpha_j}{\varepsilon}$, then for any $1 \leq s \leq j+1$

$$b_{j+1}\left(\sum_{t=1}^{s} \frac{1}{b_{t-1}} + \sum_{t=1}^{j+2-s} \frac{1}{b_{t-1}}\right) + c_{j+2} \sum_{t=1}^{j+1} \frac{1}{b_{t-1}} \geq 1 - \frac{4}{C-1}.$$

Proof of lemma

Consider graph Y, s.t. $V(Y) = V(X)$ and

$$u \sim_Y v \quad \Leftrightarrow \quad d_X(u, v) \le j + 1.$$

Take $d_X(v, w) = j + 2$, $d_X(u, v) = s$ and $d_X(w, u) = j + 2 - s$.

Triangle inequality for distinguishing numbers for u, v, w

$$\Rightarrow \lambda_s^Y + \lambda_{j+2-s}^Y \le k^Y + \mu_{j+2}^Y$$

Here $\boldsymbol{\lambda_i^Y := |N^Y(p) \cap N^Y(q)|}$ for $d_X(p, q) = i$ and $i \le j + 1$,

$\boldsymbol{\mu_{j+2}^Y := |N^Y(p) \cap N^Y(q)|}$ for $d_X(p, q) = j + 2$.

Proof of Lemma

$$\lambda_s^Y + \lambda_{j+2-s}^Y \leq k^Y + \mu_{j+2}^Y$$

Parameter λ_i^Y can be bounded with the expression that involves the number of paths that

- start at u, have length i.
- end at distance i from u.
- end at distance $\geq j+2$ from v.

Where $d_X(u, v) = j + 1$.

Proof of lemma

For $d(u, v) = j + 1$ bound the number of paths that

- start at u, have length i.
- end at distance i from u.
- end at distance $\geq j + 2$ from v.

Observe

1. At step h there are at most b_{h-1} ways to continue.
2. $\exists t \leq i$ s.t at step we enter $N_{j+2}(v)$, so $\leq b_{j+1}$ choices.

Hence the number of paths is at most

$$\prod_{h=1}^{i} b_{h-1} \sum_{t=1}^{i} \frac{b_{j+1}}{b_{t-1}}$$

Proof of lemma

Hence the number of paths is at most

$$\prod_{h=1}^{i} b_{h-1} \sum_{t=1}^{i} \frac{b_{j+1}}{b_{t-1}}$$

Therefore, inequality

$$\lambda_s^Y + \lambda_{j+2-s}^Y \leq k^Y + \mu_{j+2}^Y$$

will lead us to:

Lemma

$$b_{j+1}\left(\sum_{t=1}^{s} \frac{1}{b_{t-1}} + \sum_{t=1}^{j+2-s} \frac{1}{b_{t-1}}\right) + c_{j+2} \sum_{t=1}^{j+1} \frac{1}{b_{t-1}} \geq 1 - \frac{4}{C-1}.$$

Theorem (K. 2018)

$\forall d \geq 3$ $\exists \gamma_d > 0, m_d$, s.t. for any primitive DRG X of diameter d one of the following is true:

1. $\mathrm{mindeg}(\mathrm{Aut}(X)) \geq \gamma_d n$.
2. *X is **geometric** with **smallest eigenvalue** $-m$, where $m \leq m_d$.*

Furthermore, it is possible to take

$$m_d = \lfloor 2(d-1)(d-2)^{\log_2(d-2)} \rfloor.$$

Proof summary

Case 1: $\exists\gamma > 0$: $a_i \leq k - \gamma k \Rightarrow$ **Distinguishing number** ✓

Case 2: $\exists a_i$ close to k

$$\therefore \quad \theta_j \approx a_{\ell(j)}$$

We show: b_j large and c_{j+1} small $\implies$ b_{j+1} or c_{j+2} large.

$$\therefore \ \forall j \neq i \quad a_j \leq k - \gamma k \quad \text{for some } 0 < \gamma < 1$$

$\implies$ **Spectral gap**

Case 2a: $\lambda < \gamma k/2 \quad \implies \Omega(k)$ **Spectral tool** works.

$$\mathrm{mindeg}(X) \geq n\left(1 - \frac{q+\xi}{k}\right)$$

Case 2b: $\lambda \geq \gamma k/2 \quad \implies$ **geometric** by Metsch

DRG of diameter 3

Previous theorem says: in exceptional cases $\theta_{min} \geq -4$.

More careful analysis:

- we do not need primitivity;
- in exceptional cases $\theta_{min} \geq -3$.

Theorem (Bang, Koolen 2010)

A geometric distance-regular graph with smallest eigenvalue -3, diameter $d \geq 3$ and $\mu \geq 2$ and $n > 96$.

1. *The Hamming graph $H(3, s)$, where $s \geq 3$, or*
2. *The Johnson graph $J(s, 3)$, where $s \geq 6$.*

Case $\mu = 1$ is analyzed separately via **Bang's classification.**

DRG of diameter 3: classification

Theorem (K. 2018)

Let X be a DRG of diam. 3. Then one of the following is true.

1. $\mathrm{mindeg}(\mathrm{Aut}(X)) \geq \Omega(n)$.
2. *X is the* ***Johnson graph*** $J(s, 3)$ *for* $s \geq 7$, *or the* ***Hamming graph*** $H(3, s)$ *for* $s \geq 3$.
3. *X is the* ***cocktail-party graph****.*

DRG with smallest eigenvalue $-m$

Conjecture (Bang, Koolen 2010)

$m \geq 2$, $\forall$ geometric DRG with $\theta_{min} = -m$, diam $d \geq 3$ and $\mu \geq 2$ is

- a Johnson graph, or
- a Hamming graph, or
- a Grassmann graph, or
- a bilinear forms graph, or
- the number of vertices is bounded.

If true $\Rightarrow$ we will have **complete classification** for DRG of fixed diameter d with sublinear $\mathrm{mindeg}(\mathrm{Aut})$,
except the case $\mu = 1$.

General setup: coherent configurations

Classifications of large primitive groups

Theorem (Cameron 1982 , form of Maróti 2002)

If G is a primitive permutation group of degree $n > 24$, then one of the following is true.

1. *G is a **Cameron group**, i.e., $(A_m^{(k)})^d \leq G \leq S_m^{(k)} \wr S_d$ for some $m, k, d \in \mathbb{N}$.*
2. $|G| \leq n^{1+\log(n)}$.

Theorem (Liebeck 1984)

If G is a primitive permutation group of degree n, then one of the following is true.

1. *G is a **Cameron group***
2. $\mathrm{mindeg}(G) \geq \frac{n}{9\log_2(n)}$.

Both results depend on the CFSG.

Orbital configurations

A group $G \leq \mathrm{Sym}(\Omega)$ induce an action on $\Omega \times \Omega$.
Orbits of this action - **orbitals.**

Orbital (schurian) configuration $\mathfrak{X}(G) = (\Omega, Orbitals)$.
Observation: it is a coherent configuration.

$$G \leq \mathrm{Aut}(\mathfrak{X}(G))$$

Large primitive groups $\Rightarrow$
$\Rightarrow$ (orbital) PCC with many automorphisms

Orbital config. of Cameron groups = **Cameron schemes.**
Conjecture(Babai):

PCC with many automorphisms $\Rightarrow$ it is orbital CC for large primitive group

Conjectures for PCC (Babai) : different layering

Conjecture: Order

$\forall \varepsilon > 0$, for n large enough, and PCC $\mathfrak{X}$ on n vertices

$$|\mathrm{Aut}(\mathfrak{X})| \geq \exp(n^{\varepsilon}) \quad \Rightarrow \quad \mathfrak{X} \text{ is a Cameron scheme.}$$

Conjecture: Minimal degree

$\forall r \geq 2\ \exists \gamma_r > 0$ s.t. if $\mathfrak{X}$ is a PCC of rank r on n vertices

$$\mathrm{mindeg}(\mathrm{Aut}(\mathfrak{X})) < \gamma_r n \quad \Rightarrow \quad \mathfrak{X} \text{ is a Cameron scheme.}$$

Conjecture: Thickness

$\forall \varepsilon > 0$, for n large enough, and PCC $\mathfrak{X}$ on n vertices

$$\theta(\mathrm{Aut}(\mathfrak{X})) \geq n^{\varepsilon} \quad \Rightarrow \quad \mathfrak{X} \text{ is a Cameron scheme.}$$

Relations between considered measures

Order vs thickness:

$$|G| \geq \frac{1}{2}\theta(G)! \geq \exp[\theta(G)(\ln\theta(G) - 1)]$$

Theorem (Babai, Cameron, Pálfy 1982)

*If G is a **primitive** permutation group of degree n, then*

$$|G| = n^{O(\theta(G))} = \exp[\theta(G)O(ln(n))].$$

Large minimal degree $\Rightarrow$ small thickness

Lemma (Wielandt 1934)

$G \leq S_n$ and minimal degree satisfies $\mathrm{mindeg}(G) \geq \Omega(n)$

$$\theta(G) \leq O(\log(n)).$$

Known Progress: in terms of the order

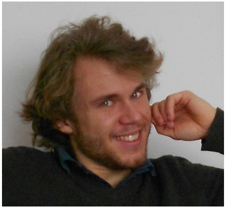
Conjecture: Order (Babai)

$\forall \varepsilon > 0$, for n large enough, and PCC $\mathfrak{X}$ on n vertices

$$|\operatorname{Aut}(\mathfrak{X})| \geq \exp(n^{\varepsilon}) \quad \Rightarrow \quad \mathfrak{X} \text{ is a Cameron scheme.}$$

Case $\varepsilon > \frac{1}{2}$: confirmed by László Babai in 1981.

Case $\varepsilon > \frac{1}{3}$: confirmed by X. Sun and J. Wilmes in 2014.



John Wilmes

Xiaorui Sun

Known Progress: In terms of the minimal degree

Conjecture: Minimal degree (Babai)

$\forall r \geq 2\ \exists \gamma_r > 0$ s.t. if $\mathfrak{X}$ is a PCC of rank r on n vertices

$$\mathrm{mindeg}(\mathrm{Aut}(\mathfrak{X})) < \gamma_r n \quad \Rightarrow \quad \mathfrak{X} \text{ is a Cameron scheme.}$$

For $r = 3$, i.e. SRG - confirmed by Babai in 2014

For $r = 4$ - confirmed in this paper

A classification of PCC of rank 4

Theorem (K. 2018)

Let $\mathfrak{X}$ be a PCC of rank 4 on n vertices . Then one of the following is true.

1. $\mathrm{mindeg}(\mathrm{Aut}(\mathfrak{X})) \geq \Omega(n)$.
2. $\mathfrak{X}$ *is a Hamming or a Johnson scheme.*

Coherent configurations of rank 4

Primitive coherent configurations of rank 4:

1. CC induced by diameter 3 DRG; ✓✓

Coherent configurations of rank 4

Primitive coherent configurations of rank 4:

1. CC induced by diameter 3 DRG; ✓✓
2. CC with two oriented constituents
   $\Rightarrow$ undirected constituent is SRG
   $\Rightarrow$ only complements to $T_2(s)$ and $L_2(s)$ are interesting. ✓

Coherent configurations of rank 4

Primitive coherent configurations of rank 4:

1. CC induced by diameter 3 DRG; ✓✓
2. CC with two oriented constituents
   $\Rightarrow$ undirected constituent is SRG
   $\Rightarrow$ only complements to $T_2(s)$ and $L_2(s)$ are interesting. ✓
3. CC with three undirected constituents of diam 2. ?
   $\Rightarrow$ Distinguishing number+spectral analysis+Metsch for $X_1, X_2, X_{1,2}$ shows that one of them is SRG or a line graph
   $\Rightarrow$ we can do more careful spectral analysis combined with Metsch
   $\Rightarrow \mathrm{mindeg}(\mathrm{Aut}(\mathfrak{X})) \geq \Omega(n)$. ✓

BIG GOAL

Conjecture (Babai)

Let $\mathfrak{X}$ be a non-Cameron PCC on n vertices. Then

1. $\theta(\mathrm{Aut}(\mathfrak{X})) \leq \log(n)^c$ (polylogarithmic)
2. $|\mathrm{Aut}(\mathfrak{X})| \leq \exp\left(\log(n)^c\right)$ (quasipolynomial)

Progress:

For part (1):

- Babai did for rank 3 (2014)
- This paper give for rank 4

For part (2):
if RHS relaxed to $\exp(n^{\varepsilon})$

- Babai in 1981 for $\varepsilon > 1/2$
- Sun, Wilmes in 2014 for $\varepsilon > 1/3$.

Related problems

Problem (Bang-Koolen conjecture)

Classify all geometric DRGs of diam d with $\theta_{min} = -m$.

Problem

Classify all geometric DRGs of diameter d with $\theta_{min} = -m$ with sublinear $\mathrm{mindeg}(\mathrm{Aut})$.

Problem

We know that all geometric DRGs with diam d and $\mathrm{mindeg}(Aut) \leq \gamma n$ *have* $\theta_{min} \geq -m_d \approx -d^{\log d}$. *Improve m_d (ideally to $m_d = d$).*

Related problems

Problem

Does there exist an infinite sequence of PCC $\mathfrak{X}$ of rank $r \geq 4$, s.t. one of the constituents is the complement to $J(n, 2)$?
If yes, what is the smallest $r \geq 4$?

Problem

Let $\mathfrak{X}$ be a PCC of small rank r. Assume $D(\mathfrak{X}) < \varepsilon n$.
Can we show "reasonable" spectral gap for at least one of the constituent graphs of $\mathfrak{X}$?