Integral Cayley graphs on Sym_n and Alt_n

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Symmetry vs Regularity

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Symmetry vs Regularity: main directions

Modern AGT includes

Association Schemes, Coherent Configurations, Cayley Graphs, GIP.

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Suppose that S is a nonempty subset of a finite group G, containing with every element its inverse, i. e. $S = S^{-1} = \{s^{-1} \mid s \in S\}$. The *Cayley graph* $\Gamma = Cay(G,S)$ of a group G associated with S is an undirected graph with the vertex set identified with G, and vertices $g, h \in G$ are joined by an edge if and only if there exists $s \in S$ such that $s = g^{-1}h$.

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Coherent closure of a Cayley graph

A coherent closure $\ll Cay(G, S) \gg$ is an association scheme.

Integral graphs: historical background

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K. Balińska, D. Cvetković, and others (1999-2000)

There are exactly 263 connected integral graphs on up to 11 vertices (based on Brendan McKay's program *GENG* for generating graphs).

Integral graphs: most graphs are nonintegral

O. Ahmadi, N. Alon, I. F. Blake, and I. E. Shparlinski, Graphs with integral spectrum, (2009)

Most graphs have nonintegral eigenvalues, more precisely, it was proved that the probability of a labeled graph on n vertices to be integral is at most $2^{-n/400}$ for a sufficiently large n.

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Question

Which Cayley graphs are integral?

Classification of integral cubic Cayley graphs

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There are exactly 7 connected cubic integral Cayley graphs. In particular, for a finite group G and a generating set $S, |S| = 3$, the Cayley graph Γ is integral if and only if G is isomorphic to one the following groups: C_2^2, C_4, C_6, Sym_3, C_2^3, $C_2 \times C_4$, D_8, $C_2 \times C_6$, D_{12}, Sym_4, Alt_4, $D_8 \times C_3$, $D_6 \times C_4$ or $A_4 \times C_2$.

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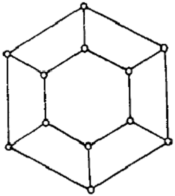
C_n is the cyclic group of order n
D_{2n} is the dihedral group of order $2n, n > 2$
Sym_n is the symmetric group of order n
Alt_n is the alternating group of order n

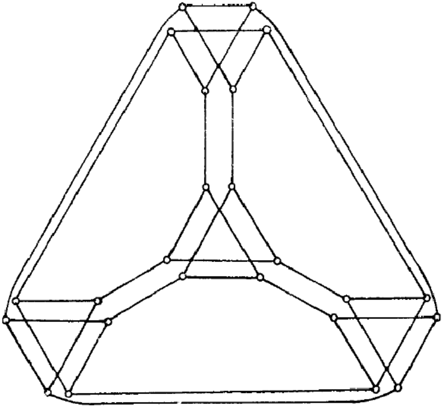
Classification of integral cubic graphs (F.C. Bussemaker, D. Cvetkovič (1976); A.J. Schwenk (1978)): Cayley graphs

G_{12}



$3,\ 2^2,\ 1,\ 0^4,\ -1,\ -2^2,\ -3$

G_{13}



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There are exactly 13 connected, cubic, integral graphs

$n = 6$	$(\pm 3, 0^4)$	
$n = 8$	$(\pm 3, (\pm 1)^3)$	
$n = 10$	$(\pm 3, \pm 2, (\pm 1)^2, 0^2)$	
$n = 12$	$(\pm 3, (\pm 2)^2, \pm 1, 0^4)$	
$n = 20$	$(\pm 3, (\pm 2)^4, (\pm 1)^5)$	
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$n = 24$	$(\pm 3, (\pm 2)^6, (\pm 1)^3, 0^4)$	Sym_4
$n = 30$	$(\pm 3, (\pm 2)^9, 0^{10})$	
$n = 4$	$(3, (-1)^3)$	
$n = 6$	$(3, 1, 0^2, (-2)^2)$	
$n = 10$	$(3, 1^5, (-2)^4)$	
$n = 10$	$(3, 2, 1^3, (-1)^2, (-2)^3)$	
$n = 12$	$(3, 2^3, 0^2, (-1)^3, (-2)^3)$	Alt_4

Integral Cayley graphs

Characterization of integral Cayley graphs

- Hamming graphs $H(n, q)$: $\lambda_m = n(q-1) - qm$, where $m = 0, 1, ..., n$, with multiplicities $\binom{n}{m}(q-1)^m$
- Cayley graphs over cyclic groups (circulants) (W. So, 2005)
- Cayley graphs over abelian groups (W. Klotz, T. Sander, 2010)
- Cayley graphs over dihedral groups (L. Lu, Q. Huang, X. Huang, 2017)

Hamming graphs are Cayley graphs

In particular, the hypercube graphs H_n (binary case of the Hamming graphs) are Cayley graph on the group $\mathbb{Z}_2^n$ with the generating set $S = \{(\underbrace{0, \ldots, 0}_{i}, 1, \underbrace{0, \ldots, 0}_{n-i-1}), 0 \leqslant i \leqslant n-1\}$.

The Star graph $S_n = Cay(\mathrm{Sym}_n, t), \ n \geqslant 2$

is the Cayley graph over the symmetric group Sym_n with the generating set $t = \{(1\ i), \ 2 \leqslant i \leqslant n\}$.

Integral Cayley graphs over Sym_n

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Properties of the Star graph

- connected bipartite $(n-1)$–regular graph of order $n!$ and diameter $diam(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ (S. B. Akers, B. Krishnamurthy (1989))
- vertex-transitive and edge-transitive
- contains hamiltonian cycles (V. Kompel'makher, V. Liskovets, 1975, P. Slater 1978)
- it does contain even ℓ–cycles where $\ell = 6, 8, \ldots, n!$ (K. Feng, A. Kanevsky 1995, J. Sheu, J. Tan, K. Chu 2006)
- has integral spectrum

Integrality of the Star graphs S_n

Conjecture (A. Abdollahi and E. Vatandoost, 2009)

The spectrum of S_n is integral, and contains all integers in the range from $-(n-1)$ up to $n-1$ (with the sole exception that when $n \leqslant 3$, zero is not an eigenvalue of S_n).

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R. Krakovski and B. Mohar proved the second part of the conjecture.

Theorem *(R. Krakovski and B. Mohar, 2012)*

Let $n \geqslant 2$, then for each integer $1 \leqslant k \leqslant n-1$ the values $\pm(n-k)$ are eigenvalues of S_n with multiplicity at least $\binom{n-2}{k-1}$. If $n \geqslant 4$, then 0 is an eigenvalue of S_n with multiplicity at least $\binom{n-1}{2}$.

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Theorem *(G. Chapuy and V. Feray, 2012)*

The spectrum of S_n contains only integers. The multiplicity $mul(n-k)$, where $1 \leqslant k \leqslant n-1$, of an integer $(n-k) \in \mathbb{Z}$ is given by:

$$mul(n-k) = \sum_{\lambda \in P(n)} dim(V_\lambda) I_\lambda(n-k),$$

where $dim(V_\lambda)$ is the dimension of an irreducible module, $I_\lambda(n-k)$ is the number of standard Young tableaux of shape λ, satisfying $c(n) = n-k$.

Integral Cayley graphs over Sym_n

J. Friedman, On Cayley graphs on the symmetric group generated by transpositions, 2000

The second smallest non-negative eigenvalue λ_2 of Cayley graphs over the symmetric group generated by transpositions was investigated.

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Theorem *(J. Friedman, 2002)*

Among all sets of $n-1$ transpositions which generate the symmetric group, the set whose associated Cayley graph has the highest $\lambda_2 = 1$ is the set $t = \{(1\ i),\ 2 \leqslant i \leqslant n\}$.

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Corollary

There are no other integral Cayley graphs over the symmetric group generated by sets of $n-1$ transpositions.

Integral Cayley graphs over Sym_n

Theorem *(D. Lytkina, E. K., 2018)*

Let $G = \mathrm{Sym}_n$ be the symmetric group of degree $n \geqslant 2$ and S be the set of all transpositions of G. Then the graph $\Gamma = Cay(G, S)$ is integral.

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Sketch of proof

- By induction over n. For $n = 2$ and $n = 3$, the statement is trivial.
- For $n > 3$, put $S = \{(ij) \mid 1 \leqslant i < j \leqslant n\}$, consider sets $T = \{(1j) \mid 2 \leqslant j \leqslant n\}$, $K = \{(ij) \mid 2 \leqslant i < j \leqslant n\}$, $T \cap K = \varnothing$, $S = T \cup K$.
- T is the set of transpositions of the stabilizer H of the symbol 1 in Sym_n. Easy to see that $T^x = T$ for every $x \in H$.
- $S_n = Cay(G, T)$ is integral. Since $\Gamma_G = Cay(G, K)$ is the union of the graphs $\Gamma_H = Cay(H, K)$, therefore it is integral by induction.
- Since $z_T = \sum\limits_{s \in T} s$ and $z_K = \sum\limits_{s \in K} s$ commute, the theorem is proved.

Integral Cayley graphs over Alt_n

Theorem *(D. Lytkina, E. K., 2018)*

Let $G = \mathrm{Alt}_n$ be the alternating group of degree $n \geqslant 2$ and let $S = \{(1ij) \mid 2 \leqslant i,j \leqslant n, i \neq j\}$ be the set of 3-transpositions of G. Then the graph $\Gamma = Cay(G,S)$ is integral and its spectrum coincides with the set

$$\{-n+1, 1-n+1, 2^2-n+1, \ldots, (n-1)^2-n+1\}$$

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Open question

What are multiplicities of the eigenvalues above?

Other open questions

Problem 19.50 b), The Kourovka Notebook, 2018,
arXiv : 1401.0300

Let Alt_n be the alternating group of degree n and $S = R \bigcup R^{-1}$ where $R = \{(123), (124), \ldots, (12n)\}$. Is it true that the Cayley graph $\Gamma = Cay(Alt_n, S)$ is integral?

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Problem 19.50 a): Yes, it is true! (D. O. Revin, A. Abdollahi)

Both proofs suggested are based on character theory.

THANK YOU!