Invariants for efficiently computing the autotopism group of a partial Latin rectangle

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joint work with

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Symmetry vs Regularity, Pilsen, 2018

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2	·	·	4	1	5	6	·	
·	1	5	3	·	4	·	·	·
·	2	·	5	·	3	·	4	·
4	3	·	·	5	·	1	·	2
·	·	·	·	2	·	·	1	3

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The entry set of L

$$\mathrm{Ent}(L) := \{(i, j, L[i, j]) : i \in [r], j \in [s], L[i, j] \in [n]\}$$

Isotopisms and autotopisms

$\Theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$ an isotopism

$\Theta : \mathrm{PLR}(r, s, n) \rightarrow \mathrm{PLR}(r, s, n)$

α permutes the rows

β permutes the columns

γ permutes the symbols

If $\Theta(L) = L$ then Θ is an autotopism of L

$\mathrm{Atop}(L) =$ The autotopy group of L

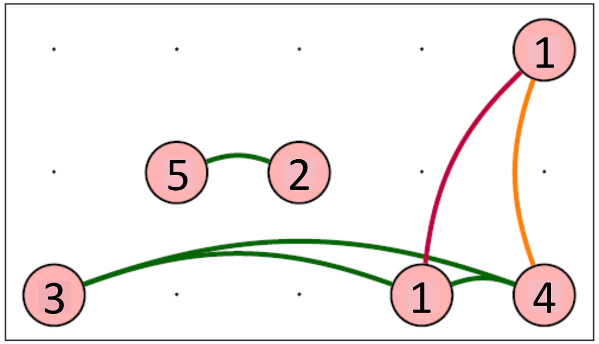
Computing $\mathrm{Atop}(L)$

The partial Latin rectangle graph

(Falcon and Stones, Disc. Math. 2017)

Example:

·	·	·	·	1
·	5	2	·	·
3	·	·	1	4



Compute $\mathrm{Atop}(L)$ by computing graph automorphisms.

Use ‘nauty’ (McKay, Meynert, Myrvold, JCD 2007) and its variants

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Example:

the types system $\mathfrak{P}_{\mathrm{T}} = \left(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\right)$:

$\mathcal{P}_1$ is defined by the number of entries in the rows

$\mathcal{P}_2$ is defined by the number of entries in the columns

$\mathcal{P}_3$ is defined by the number of appearances of each symbol

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How?

1) Start with $\mathfrak{P} = \mathfrak{P}_S(L)$ the trivial system of partitions
2) Apply refinement methods

The natural refinement

$\mathfrak{P} = \left(\mathcal{P}_{\mathrm{row}}, \mathcal{P}_{\mathrm{col}}, \mathcal{P}_{\mathrm{sym}}\right)$ induces a partition $E\left(\mathfrak{P}\right)$ on $\mathrm{Ent}(L)$:

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$$(i, j, L[i, j])) \sim_{E(\mathfrak{P})} (i', j', L[i', j']) \Leftrightarrow \begin{cases} i \sim_{\mathcal{P}_{\text{row}}} i' \\ j \sim_{\mathcal{P}_{\text{col}}} j' \\ L[i, j] \sim_{\mathcal{P}_{\text{sym}}} L[i', j'] \end{cases}$$

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1) Label the elements of $\mathrm{Ent}(L)$ by their part in $E(\mathfrak{P})$

2) Define the natural refinement $N(\mathfrak{P}) = \left(N(\mathcal{P}_{\text{row}}), N(\mathcal{P}_{\text{col}}), N(\mathcal{P}_{\text{sym}})\right)$ where

$N(\mathcal{P}_{\text{row}})$ is define by the multisets of labels in the rows

$N(\mathcal{P}_{\text{col}})$ is define by the multisets of labels in the columns

$N(\mathcal{P}_{\text{col}})$ is define by the multisets of labels corresponding to the symbols

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Examples: $N(\mathfrak{P}_\mathrm{S}) = \mathfrak{P}_\mathrm{T}$ $\quad N(N(\mathfrak{P}_\mathrm{S})) = \mathfrak{P}_\mathrm{SEI}$

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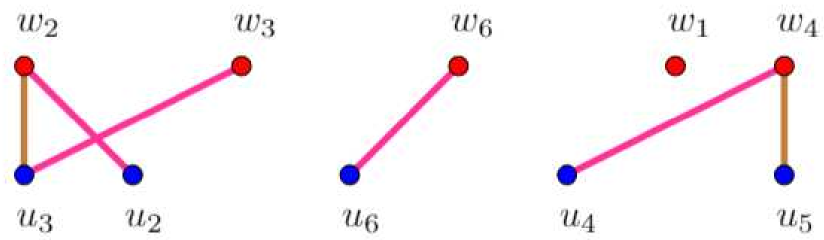
Disadvantage: when starting from $\mathfrak{P}_{\mathrm{S}}$ it is useless for very dense PLRs

Two-line graphs

for two rows r_1, r_2 in a PLR, construct a vertex-and-edge-colored bipartite graph $G_{r_1,r_2}(L)$ as illustrated here:

$$L = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 2 & \cdot & 3 & \cdot & 6 & \cdot & 1 & 4 \\ \hline 3 & 2 & \cdot & 6 & \cdot & 4 & \cdot & 5 \\ \hline \end{array} \Rightarrow$$



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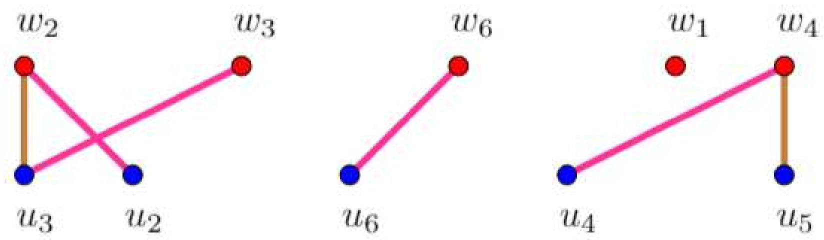

Similar constructions for two columns and two symbols.

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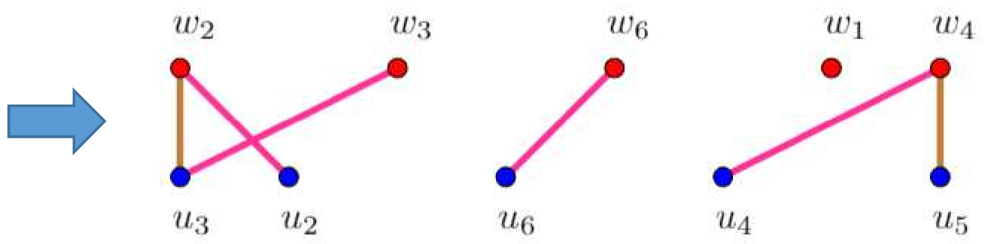

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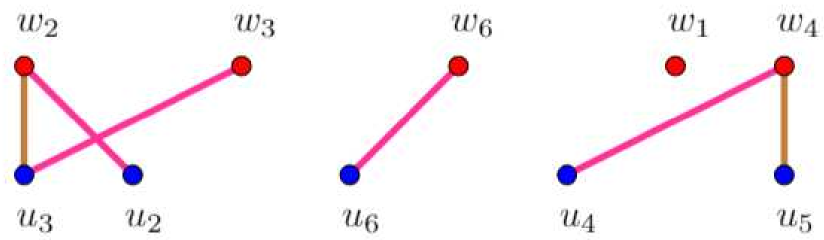

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Remark: can be viewed as a generalization of cycles in partial permutations

The two-line representations

$$\mathcal{R}_{\text{row}}(L) = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & r_{kl} & \\ & & r_{ij} & & \\ & & & & 0 \end{bmatrix}$$

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(Define $\mathcal{R}_{\mathrm{col}}(L)$ and $\mathcal{R}_{\mathrm{sym}}(L)$ analogously)

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$$r_1 \sim_{G(\mathcal{P}_{\mathrm{r}})} r_2 \quad \Leftrightarrow \quad \begin{cases} r_1 \sim_{\mathcal{P}_{\mathrm{r}}} r_2 \\ \text{the multisets agree in each part} \end{cases}$$

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$$\mathcal{R}_{\mathrm{row}}(L) = \begin{array}{c|c|c|c|c|c} & P_1 & P_2 & P_i & & P_k \\ r_1 & 1\ 1\ 3\ 2 & 4\ 2\ 2 & \cdots & & 6\ \ 12\ \ 9 \\ r_2 & 3\ 1\ 1\ 2 & 2\ 4\ 2 & & & 12\ \ 9\ \ 6 \end{array}$$

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analogous definitions for $G(\mathcal{P}_{\mathrm{c}})$ and $G(\mathcal{P}_{\mathrm{s}})$

$$G(\mathfrak{P}) = \big(G(\mathcal{P}_{\mathrm{r}}), G(\mathcal{P}_{\mathrm{c}}), G(\mathcal{P}_{s})\big)$$

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$$\Downarrow$$

$$G\left[\mathfrak{P}_S\right] \leq N\left[\mathfrak{P}_S\right]$$

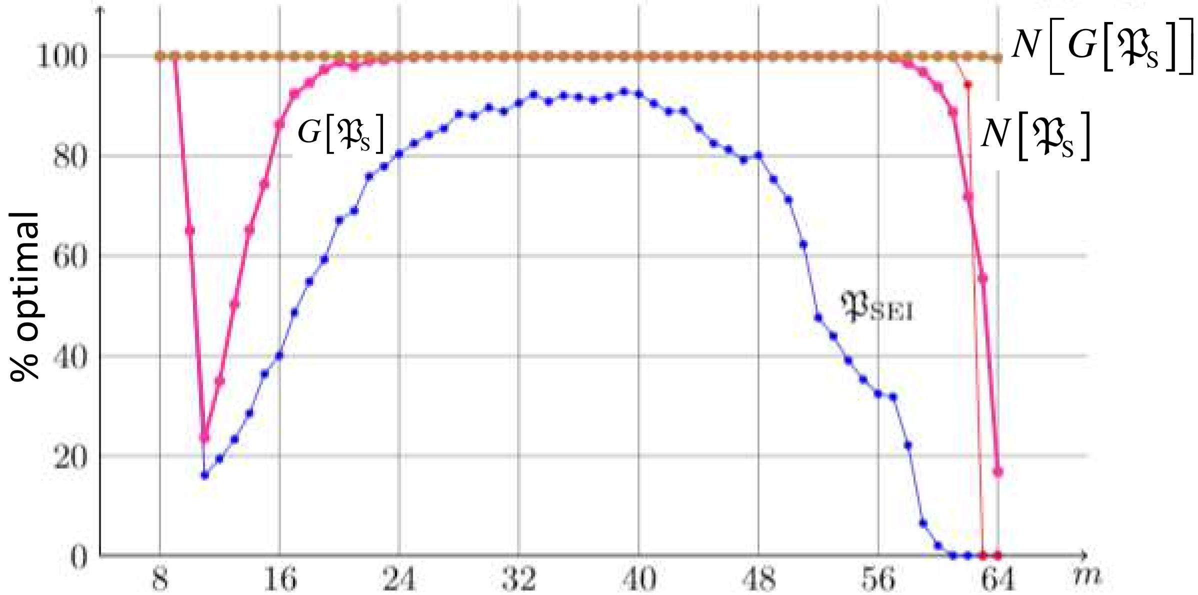
This does not mean that the natural refinement is redundant. It is possible to have

$$N\left(G[\mathfrak{P}]\right) < G[\mathfrak{P}]$$

Complexity

The average complexity of the TLG refinement for $L \in \mathrm{PLR}(r, s, n)$ is $\mathcal{O}(M^3 \log M)$, where $M = \max(r, s, n)$

Performance on random PLRs



1000 random PLRs in PLR(8,8,8;m) for each m

Works well for dense PLRs and (full) Latin rectangles.

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- Find other refinement methods.
- What is the most efficient way to conduct the subsequent search?

T	•	H	•	A	•	N	K	•
•	•	•	Y	O	•	U	•	
•	F	•	O	•	R	•	•	•
•	Y	O	•	•	U	•	R	•
•	T	•	•	•	•	•	•	•
A	•	T	E	N	•	I	O	N
•	•	•	•	•	T	•	•	•