On two-fold orbitals

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Symmetry vs Regularity
Pilsen, 1-7 July 2018

Two-fold permutations

A two-fold permutation group of V is a subgroup Γ of $S_V \times S_V$ whose action on $V \times V$ is defined by:

$$(\alpha, \beta) : (u, v) \mapsto (u^\alpha, v^\beta).$$

One application / motivation

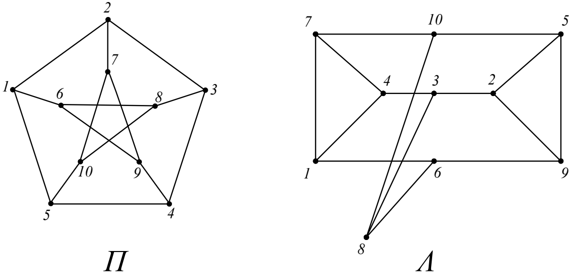
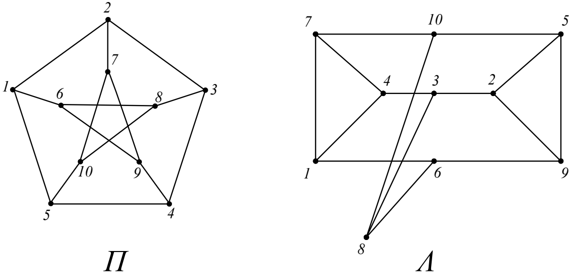


Figure: The Petersen graph and the Livio Porcu graph — they are not determined by their neighbourhoods

Why?

One application / motivation



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Two-fold isomorphisms and automorphisms

Two (mixed) graphs G and H are said to be *two-fold isomorphic* or TF-isomorphic if there exist bijections α and β from V(G) to V(H) such that (u,v) is an arc of G if and only if (u^α, v^β) is an arc of H. If $G = H$ then we say that (α, β) is a two-fold automorphism of G.

Note that we need to consider every edge $\{u,v\}$ of G as the union of the two arcs (u,v) and (v,u) since the images of these two arcs are, in general, not opposite arcs under the action of (α, β).

Two-fold isomorphisms and automorphisms

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If α is an automorphism of G then (α, α) can be considered to be a TF-automorphism of G.

We call those TF-automorphisms (α, β) for which $\alpha \neq \beta$ *non-trivial* TF-automorphisms of G.

The set of all TF-automorphisms of G is a group under componentwise multiplication, and we denote this group by $\mathrm{Aut}^{\mathrm{TF}}(G)$.

Clearly, if we consider (α, α) to be a TF-automorphism of G, then $\mathrm{Aut}(G)$ is a subgroup of $\mathrm{Aut}^{\mathrm{TF}}(G)$ and this inclusion is strict if G has non-trivial TF-automorphisms.

A graph which has a non-trivial TF-automorphism is said to be *unstable*.

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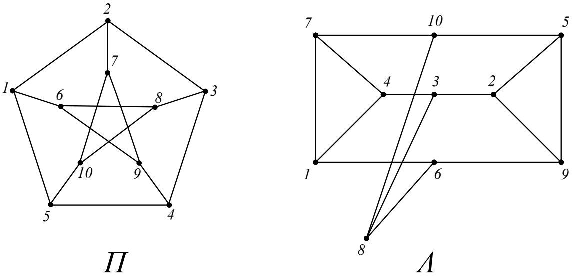
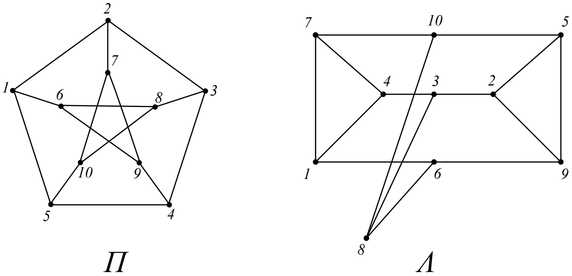


Figure: The Petersen graph and the Livio Porcu graph — they are not determined by their neighbourhoods

Why?

Because they are TF-isomorphic!

And for this reason they also have the same canonical double cover.



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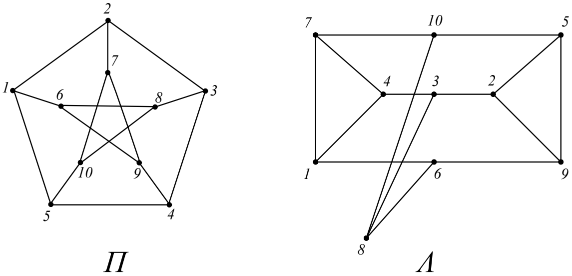

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One result about neighbourhood reconstruction

Theorem

Let G be a connected bipartite graph. Then G is not reconstructible from its family of neighbourhoods iff its automorphism group has an involution which switches its colour classes but does not fix an edge.

TF-isomorphisms give an easy proof.

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Can an asymmetric graph have non-trivial TF-automorphsms (hidden symmetries)?

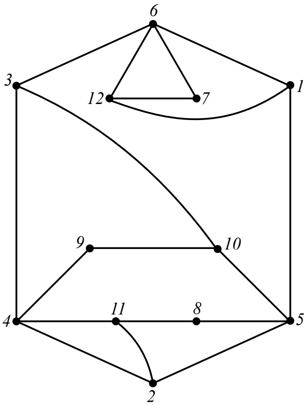
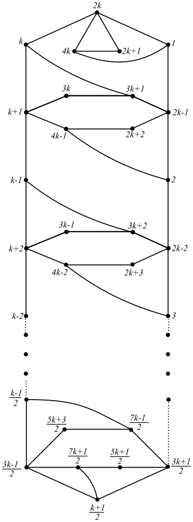


Figure: This is the smallest asymmetric graph with non-trivial TF-automorphisms

A family of asymmetric graphs with arbitrarily large $(\geq k-1)$ number of TF-automorphisms



Two-fold orbitals

Let Γ be a TF-permutation group acting on $V \times V$. A TF-orbital of Γ is an orbit of the action of Γ on $V \times V$.

The figure shows an example the two-fold orbitals of a two-fold permutation group.

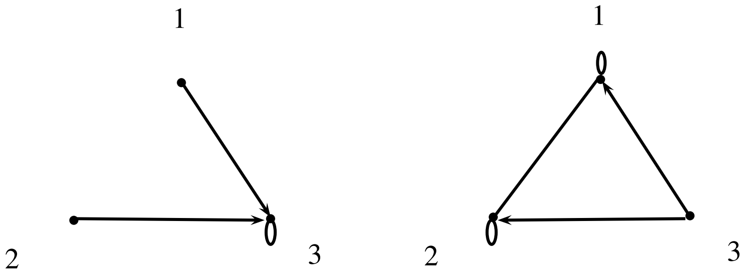


Figure: TF-orbitals of $\Gamma = \langle((1,2,3),(1,2))\rangle$

TF-rank equal to 1

We know, in general, that the number of orbitals of a permutation group (Γ, V) is at least 2 and this happens only when (Γ, V) is 2-transitive.

However, unlike the usual rank, the TF-rank can be equal to 1.

This is possible because TF-permutations can take arcs to loops.

The following result characterizes the actions whose TF-rank is equal to 1.

TF-transitivity and Σ-transitivity

Let $\Gamma \leq S_V \times S_V$ be a two-fold permutation group acting on the set $V \times V$. We say that Γ is Σ-transitive on V if for any $u, v \in V$, there exists $(\alpha, \beta) \in \Gamma$ such that $u^\alpha = v^\beta$.

We also say that Γ is TF-transitive on V if, for all $u, v \in V$, there exists $(\alpha, \beta) \in \Gamma$ such that $u^\alpha = v$ *and* $u^\beta = v$.

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Characterisation of TF-rank equal to 1

Theorem

Let $\Gamma \subseteq S_V \times S_V$ *be a two-fold permutation group. Then,* $(\Gamma, V \times V)$ *has TF-rank equal to* 1 *if and only if* Γ *is both* Σ*-transitive and TF-transitive on* V*.*

Structure constants of TF-orbitals

In general, colour graphs arising from TF-orbitals do not admit structure constants. For example,

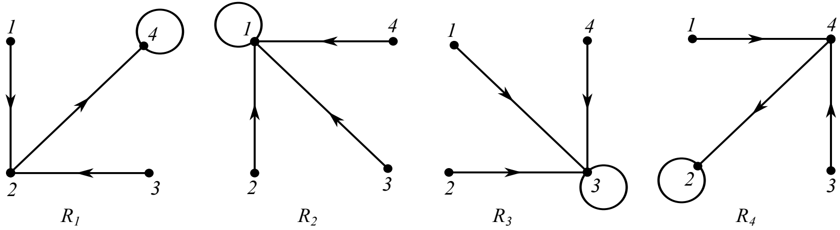

Figure: The TF-orbitals for $\Gamma = \langle(\alpha, \beta)\rangle$ where $\alpha = (1\ 2\ 3\ 4)$ and $\beta = (2\ 4)$.

Structure constants of TF-orbitals (2)

On the other hand, there exist systems of TF-orbitals that admit structure constants. For example,

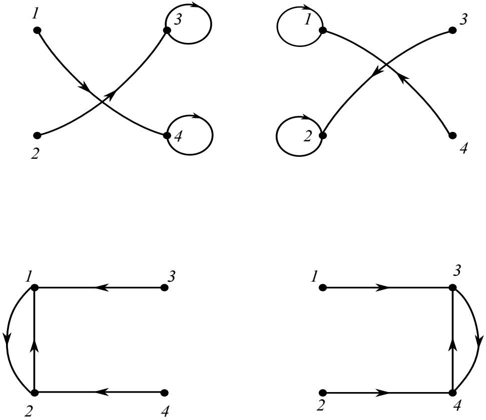

Figure: The system of TF-orbitals for $\Gamma = \langle ((1\ 2\ 3\ 4), (1\ 2)(3\ 4)) \rangle$.

Sufficient conditions for structure constants

A two-fold permutation group Γ is said to satisfy *Property K* if, for any $x, y \in V$ and any $(\alpha, \beta) \in \Gamma$, the arcs (x, y) and (x^β, y^β) are in the same TF-orbital.

Theorem

Suppose $\Gamma \leq S_V \times S_V$ satisfies Property K. Then, given any arc (a, b) in the TF-orbital R_k, the number of vertices x such that (a, x) is in R_i and (x, b) is in R_j is independent of the choice of (a, b) in R_k. Therefore the TF-orbitals admit the definition of structure constants p_{ij}^k.

Sufficient conditions (2)

A two-fold permutation group Γ is said to satisfy *Property M* if, for any (α, β) in Γ, (β, β) is also in Γ.

Property M implies Property K but the converse does not hold in general.

The fact that Property M implies Property K makes it easier to obtain two-fold permutation groups fulfilling Property K.

Moreover, if it is also true that even (α, α) is in Γ, then the TF-orbitals are closed under taking of transpose.

NOTE: It is also true that if all non-trivial TF-permutations are of the form (α, α^{-1}) *then the TF-orbitals are undirected graphs.*

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Directed alternating walks of length 3

Although TF-orbitals do not in general admit structure constants, it is easy to prove that an extension of the structure constants to directed alternating walks of length 3 can, in general, be defined.

Theorem

Let $\Gamma \leq S_V \times S_V$ and let $R_1, R_2, \ldots, R_r$ be the TF-orbitals of Γ. Let i, j, k and s be any elements of $\{1, 2, \ldots, r\}$. Let (a,b) be an arc in R_s. Then the number of arcs (y,x), such that $(y,x) \in R_j$, $(a,x) \in R_i$ and $(y,b) \in R_k$ is independent of the choice of arc (a,b) in R_s.

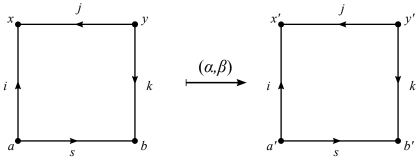

Figure: Definition of p^s_{ijk} for TF-orbitals.

This result can be can be expressed in terms of the adjacency matrices of the R_i as:

$$A_i A_j^T A_k = \Sigma_{t=1}^{r} p^s_{ijk} A_s.$$

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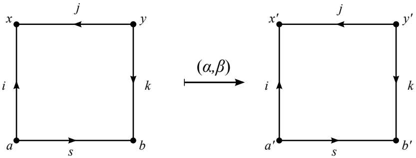

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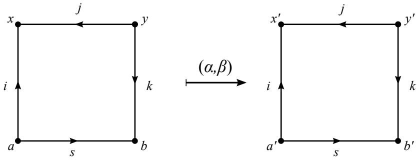

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Simple application to rank 3 unstable SRGs

Alternative proof of a result of Surowski obtained by using

$$A_iA_j^TA_k \;=\; \Sigma_{t=1}^{r} p_{ijk}^{s} A_s.$$

Theorem

Let G be a rank 3 unstable strongly regular graph with parameters n, k, λ, μ. Then $\lambda = \mu$.

An example not coming from TF-orbitals (M. Klin, Novy Smokovec lectures)

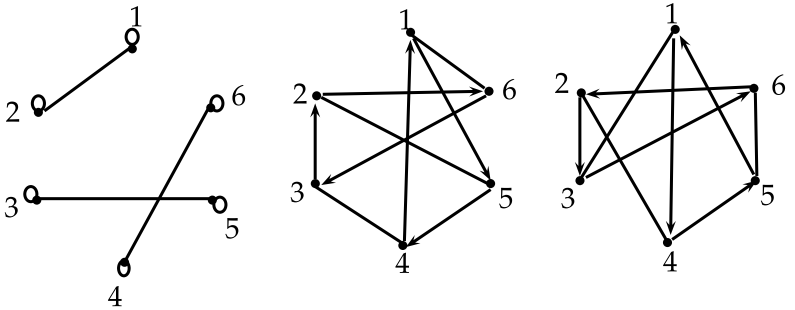

Figure: Adjacency matrices obtained by bringing together the permutation matrices of the regular action action of S_3 into three disjoint subsets and adding them. Orbitals admit structure constants.

The corresponding adjacency matrices are

$$A_0 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Conditions satisfied / not satisfied by these colour graphs

The space generated by their adjacency matrices is:

1. Closed under matrix multiplication.
2. Closed under SH-multiplication.
3. Does not contain the identity.
4. Might not be closed under taking of transpose.
5. If the colour graphs are TF-orbitals they also satisfy

$$A_i A_j^T A_k = \Sigma_{t=1}^{r} p_{ijk}^{s} A_s.$$

Thank you!