Inverse problems in spectral theory of graphs

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Symmetry vs Regularity

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Introduction

Let G be a simple finite graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G)$.

With graph G we asociate adjacency matrix $A(G) = (a_{ij})_{i,j=1}^{n}$

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of G is determined as

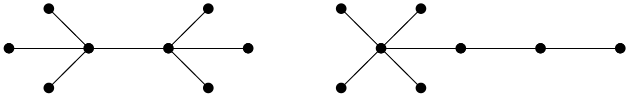
$$P_G(\lambda) = det(A(G) - \lambda I).$$

Two graphs G and H are cospectral, if

$$P_G(\lambda) = P_H(\lambda).$$

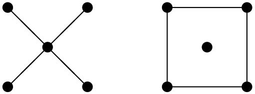
Cospectral graphs with respect to

Adjacency matrix



$$\sigma(A) = \{0^4, \frac{1\pm\sqrt{13}}{2}, \frac{-1\pm\sqrt{13}}{2}\}$$

Figure 1: A pair of cospectral trees (1957, L. Collatz and U. Sinogowitz)



$$\sigma(A) = \{0^3, \pm 2\}$$

Figure 2: Cospectral graphs with respect to A, "Saltire pair"(1971, D. Cvetkovich)

n	*Numbers of graphs*	A
1	1	0
2	2	0
3	4	0
4	11	0
5	34	2
6	156	10
7	1044	110
8	12346	1722
9	274668	51039
10	12005168	2560606
11	1018997864	215331676
12	165091172592	31067572481

Table 1: Numbers of graphs with cospectral mates (2009, A. E. Brouwer, E. Spence)

Generalized adjacency matrix

Define the set of generalized adjacency matrices

$$\mathcal{A} = \{A + \alpha J | \alpha \in \mathbb{R}\},$$

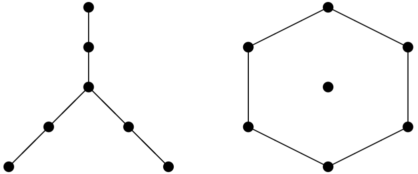
where A is adjacency matrix, J is the all-ones matrix.

Two graphs G and H are cospectral w.r.t. generalized adjacency matrix, if

$$P_G(\lambda) = P_H(\lambda)$$

and

$$P_{\overline{G}}(\lambda) = P_{\overline{H}}(\lambda).$$



$$\sigma(A) = \{-1^2, 0, 1^2, \pm 2\}$$

$$\sigma(\overline{A}) = \{-2^2, 0^2, 1, \frac{\pm\sqrt{33}+3}{2}\}$$

Figure 3: Cospectral graphs with respect to $\mathcal{A}$.

n	*Numbers of graphs*	$\mathcal{A}$
1	1	0
2	2	0
3	4	0
4	11	0
5	34	0
6	156	0
7	1044	40
8	12346	1160
9	274668	43947
10	12005168	2413039
11	1018997864	211951556

Table 2: Numbers of graphs with cospectral mates (2003, E. R. van Dam, W. H. Haemers)

GM-switching

Theorem (Godsil, McKay (1982))

Let N be a (0,1)- matrix of size $b \times c$ whose column sums are $0, b$ or $\frac{b}{2}$. Define $\tilde{N}$ to be a matrix obtained from N by replacing each column v with $\frac{b}{2}$ ones by its complememnt $1 - v$. Let B be a symmetric $b \times b$ matrix with constant row (and column) sums, and let C be a symmetric $c \times c$ matrix.

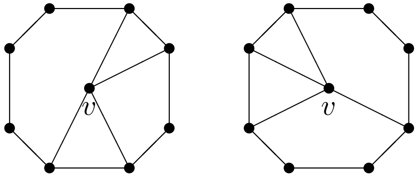
Put $M = \begin{pmatrix} B & N \\ N^T & C \end{pmatrix}$ $\tilde{M} = \begin{pmatrix} B & \tilde{N} \\ \tilde{N}^T & C \end{pmatrix}$.

Then matrices M and $\tilde{M}$ are cospectral.

Example. We consider GM-switching for graph G taking the cycle C_{2n} and adjoining a vertex v adjacent to half the vertices of C_{2n}.

For $n \leq 3$ there are no pairs of cospectral graphs.

For $n = 4$ there is one pair of cospectral graphs.



$$P_G(\lambda) = \lambda^9 - 12\lambda^7 - 4\lambda^6 + 42\lambda^5 + 16\lambda^4 - 52\lambda^3 - 16\lambda^2 + 16\lambda$$

$$P_{\overline{G}}(\lambda) = \lambda^9 - 24\lambda^7 - 40\lambda^6 + 52\lambda^5 + 94\lambda^4 - 48\lambda^3 - 56\lambda^2 + 19\lambda + 2$$

Figure 4: Cospectral graphs (C_8, v)

For $n = 5$ there are three pairs of cospectral graphs.

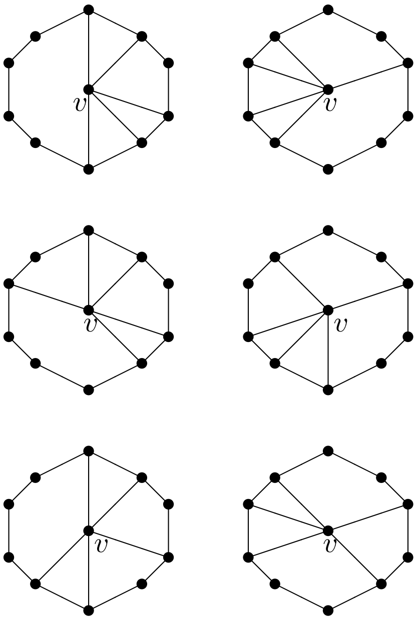


Figure 5: Cospectral graphs (C_{10}, v)

References

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[2] D.M. Cvetkovich, M. Doob, H. Sachs *Spectra of Graphs* // third ed., Johann Abrosius Barth Verlag, 1995.

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Thank you!