On packings of disjoint copies of the Hoffman-Singleton graph into K_{50}

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Introduction

General notes

Decomposition = edge decomposition of a (usually complete) graph Γ into factors.

k-packing of Σ = a decomposition where k factors are isomorphic to Σ.

Automorphism group = automorphism group of the corresponding color graph.

Petersen graph (Pet) = $SRG(10, 3, 0, 1)$, $(3, 2)$-Moore graph, $(3, 5)$-cage.

Hoffman-Singleton graph (HoSi) = $SRG(50, 7, 0, 1)$, $(7, 2)$-Moore graph, $(7, 5)$-cage.

History

Pet and K_{10}

A. J Schwenk (1983): Can the 45 edges of the complete graph K_{10} be partitioned into three copies of the Petersen graph?

Ho-Si and K_{50}

A. Rosa (2002): Does the complete graph K_{50} decompose into seven copies of the HoffmanSingleton graph?

E. van Dam (2003): Does there exist a non-trivial decomposition of K_{50} into (primitive) strongly regular graphs?

J. Šiagiová, M. Meszka (2003): For which k there exists a k-packing of HoSis into K_{50}?

K_{10}: Solutions

J. Bosák, A. Rosa, Š Znám (1966): K_{10} cannot be decomposed into three factors of diameter two.

??: There exists unique 2-packing of Pet into K_{10}, its automorphism group has size 5.

K_{50}: Known results

E. van Dam (2003): A strongly regular decomposition of $K_5$0 into three graphs does not exist.

J. Šiagiová, M. Meszka (2003): There exist packings of 5 copies of Hoffman-Singleton graph into K_{50}.

All packings found by Šiagiová and Meszka have automorphism group of size 25. In fact, they were looking only for packings with automorphism group of size 25.

We can use the method of Šiagiová and Meszka for other groups of automorphisms and also for the problem of existence of strongly regular decompositions of K_{50}.

The problem

Find

- a strongly regular decomposition of K_{50}, or

- a 6 packing of HoSi into K_{50}

with non-trivial automorphism group.

Strongly regular decompositions of K_{50}

7×HoSi,

4×HoSi and $SRG(50, 21, 8, 9)$,

3×HoSi and $SRG(50, 28, 15, 16)$

(by E. van Dam HoSi and $2 \times SRG(50, 21, 8, 9)$ does not exist).

Tools of the trade

Equitable partitions and collapsed matrices

Let Γ be a graph. A partition $\{V_1, \ldots, V_l\}$ of $V(\Gamma)$ is equitable iff

$$\forall i, j \forall u, v \in V_i : \ |N(u) \cap V_j| = |N(v) \cap V_j|.$$

Orbits of a group of automorphisms of Γ form an equitable partition.

Let $\{v_1, \ldots, v_l\}$ be a system of representatives of equitable partition $P = \{V_1, \ldots, V_l\}$. Collapsed matrix of P is matrix

$$(|N(v_i) \cap V_j|)_{l \times l}.$$

Properties of equitable partitions

Let $\{\Sigma_1, \ldots, \Sigma_m\}$ be a decomposition of Γ. Let P be a partition of $V(\Gamma)$ which is equitable for all Σ_i's and Γ, with collapsed matrices $B_1, \ldots, B_m$ and B, respectively. Then

$$B_1 + B_2 + \cdots + B_m = B.$$

We will refer to a solution of the above equation as decomposition of the collapsed matrix of Γ.

Voltage assignments and their lifts

Introduced in topological graph theory in 1970's.

Let G be a (finite) group. A voltage assignment (with values in G) is an $n \times n$ matrix

$$\alpha = (\alpha_{ij})$$

such that $\alpha_{ij} \subseteq G$.

Lift $L(\alpha)$ of a voltage assignment α is the digraph with vertex set $G \times \{1, \ldots, n\}$ and dart set defined by:

$$\forall i, j \forall x \in \alpha_{ij} : \; (g, i) \sim (xg, j).$$

Lift $L(\alpha)$ is a simple graph iff $\alpha_{ij} = \alpha_{ji}^{-1}$ and $e_G \notin \alpha_{ii}$.

For $n = 1$ we obtain Cayley connection sets and Cayley graphs.

Lifts and automorphisms

G acts semi-regularly on $L(\alpha)$ by right multiplication as a group of automorphisms. Orbits are $G \times \{i\}$ and the collapsed matrix is $(|\alpha_{ij})$.

Any digraph Γ on which G acts semi-regularly as a group of automorphisms is isomorphic to some $L(\alpha)$.

All of the above can be easily generalized to automorphism groups with orbits with sizes 1 and G.

Automorphisms of HoSi

Automorphism of prime order. Fix is the subgraph induced on the set of fixed points and K is the natural bound on the size of k-packing with given automorphism implied by Fix.

ord	Fix	K
2	$K_{1,5}$	1
2	Pet	2
3	$K_{1,4}$	1
5	K_5	2
5*	$\emptyset$	
7	K_1	

$*$- there are two nonequivalent actions.

Plan of attack

Pick your favorite automorphism of HoSi.

Find decompositions of collapsed matrix of K_{50}.

For a given decomposition find all possible systems of edge disjoint HoSis.

If necessary, check the properties of the remaining graph.

Note, that we know all the voltage assignments which lift to HoSi.

Results

Order 7

One action.

420 collapsed matrices.

No strongly regular decompositions of collapsed matrix of K_{50}.

Exactly 4795 decompositions of collapsed matrix of K_{50} which may be lifted to a 6-packing of HoSi.

Order 5

Two actions.

$126 + 3622880$ collapsed matrices.

No strongly regular decompositions of collapsed matrix of K_{50}.

More than 500000 decompositions of collapsed matrix of K_{50} which may be lifted to a 6-packing of HoSi. (This is just a by-product of our search for 7-packings. The total amount of solutions may be significantly larger).

Main results

1. Any strongly regular decomposition of K_{50} has trivial (color-preserving) automorphism group.

2. There exist exactly 1602 6 packings of HoSi with an automorphism of order 7.

Statistics

We have 1602 6 packings of HoSi. The last graph will be called remainder.

Remainders are all different.

There are no co-spectral remainders.

Only 1595 remainders are connected.

Statistics

No connected remainder is bipartite.

$Diam$	$Girth$	$\|Aut\|$	$Nr.$
3	3	7	1480
3	3	21	9
3	4	7	1
4	3	7	105

Statistics

In each of the disconnected remainders there is one component of size 42 and one component of size 8 (inducing K_8). We present properties of graphs induced by components of size 42. As the second component is K_8, the automorphism groups of whole remainders are 40320 times larger.

$Diam$	$Girth$	$\lvert Aut\rvert$	$Nr.$
3	3	7	2
3	3	42	1
3	3	84	1
3	4	84	1
3	4	84	1
4	3	84	1

Statistics

In one case the component of size 42 is bipartite. All graphs with automorphism group of order 84 are vertex transitive, the graph with automorphism group of order 42 is not vertex transitive. There are two non-isomorphic abstract groups of order 84 which are automorphism groups.

Time analysis

Programmed in GAP.

Decomposition of collapsed matrices: cca 1 CPU week (mostly for order 5).

Search for voltage assignments: cca 4 CPU years (6 computers with 4 years old i5 proc., 4 threads per proc., 2 months of computations).

Thank You