On pronormal subgroups in finite groups

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Definitions

Agreement. We consider finite groups only.

A subgroup H of a group G is *pronormal* (Ph. Hall, 1960s) in G if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Denotation: $H\, prn\, G$.

Well-known examples of pronormal subgroups:

- normal subgroups;
- maximal subgroups;
- Sylow subgroups.

Theorem* (Ph. Hall, 1960s). *Let G be a group and $H \leq G$. H is pronormal in G if and only if in any transitive permutation representation of G, the subgroup $N_G(H)$ acts transitively on the set $fix(H)$ of fixed points of H.*

A group G is called a *CI-group* (L. Babai, 1977) if between every two isomorphic relational structures on G (as underlying set) which are invariant under the group $G_R = \{g_R \mid g \in G\}$ of right multiplications

$$g_R : x \mapsto xg,$$

there exists an isomorphism which is at the same time an automorphism of G.

Pronormality works...

Theorem (L. Babai, 1977). *G is a CI-group iff G_R is pronormal in $Sym(G)$.*

Corollary. *If G is a CI-group, then G is abelian.*

Proof. The regular subgroups G_R and $G_L = \{g_L \mid g \in G\}$, where

$$g_L : x \mapsto g^{-1}x,$$

are conjugate in $Sym(G)$ and $[G_R, G_L] = 1$ in view of the associativity of multiplication. Therefore, if G is nonabelian, then $G_R \neq G_L$ and G_R and G_L are not conjugate in $\langle G_R, G_L \rangle$.

Theorem (P. Pálfy, 1987). *G is a CI-group iff $|G| = 4$ or G is cyclic of order n such that $(n, \varphi(n)) = 1$.*

General Problem

General Problem. *Let $H \leq G$, is H pronormal in G?*

General Idea. *Try to reduce to groups of smaller orders.*

Proposition 1 (Frattini's Argument). *Let $A \trianglelefteq G$, and $H \leq A$. Then the following statements are equivalent:*
(1) H is pronormal in G;
(2) H is pronormal in A and $G = AN_G(H)$.

Proposition 2. *Let $H \leq G$ and $N \trianglelefteq G$, and let $^{-}: G \to G/N$ be the natural epimorphism. The following statements hold:*
(1) if H is pronormal in G, then $\overline{H}$ is pronormal in $\overline{G}$;
(2) H is pronormal in G if and only if $\overline{HN}$ is pronormal in $\overline{G}$ and H is pronormal in $N_G(HN)$;
(3) if $N \leq H$ and $\overline{H}$ is pronormal in $\overline{G}$, then H is pronormal in G.

Reductions

Let G be non-simple and A be a minimal non-trivial normal subgroup of G. One of the following cases arises:

Case 1: $A \leq H$. $H\, prn\, G$ iff $H/A\, prn\, G/A$ (Proposition 2).

Note that $|G/A| < |G|$.

Case 2: $H \leq A$. $H\, prn\, G$ iff $H\, prn\, A$ and $G = AN_G(H)$ (Proposition 1).

Note that $|A| < |G|$.

Case 3: $H \not\leq A$ and $A \not\leq H$. If $N = N_G(HA)$, then $H\, prn\, G$ iff $HA/A\, prn\, G/A$, $N = AN_N(H)$, and $H\, prn\, HA$ (Propositions 1 and 2).

With using inductive reasonings, it is possible to reduce this case to the subcase when $G = HA$.

Theorem (Ch. Praeger, 1984). *Let G be a transitive permutation group on a set Ω of n points, and let K be a non-trivial pronormal subgroup of G. Suppose that K fixes exactly f points of Ω. Then*
(1) $f \leq \frac{1}{2}(n-1)$, and
(2) if $f = \frac{1}{2}(n-1)$, then K is transitive on its support in Ω, and either $G \geq Alt(n)$, or $G = GL_d(2)$ acting on the $n = 2^d - 1$ non-zero vectors, and K is the pointwise stabilizer of a hyperplane.

If in some transitive permutation representation of G, $fix(H)$ is too big, then H is not pronormal in G.

Natural restriction. *There is $S \leq H$ such that $S \, prn \, G$.*

Lemma*. *Suppose that G is a group and $H \leq G$. Assume also that H contains a Sylow subgroup S of G. Then the following statements are equivalent:*
(1) H is pronormal in G;
(2) The subgroups H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.

Remark 1. *If $S \in Syl_p(G)$ and $N_G(S) = S$, then for each $H \leq G$, if $|G : H|$ is not divisible by p, then $H \, prn \, G$.*

Remark 2. *Sylow subgroups of odd orders are never self-normalized in nonabelian simple groups (G. Glauberman, J. Thompson, R. Guralnick, G. Malle, G. Navarro, finished in* 2003), *while Sylow* 2*-subgroups are "usually" self-normalized in nonabelian simple groups (A. Kondrat'ev,* 2005).

Let G be non-simple, $H \leq G$, A be a minimal non-trivial normal subgroup of G, $G = HA$, and $|G : H|$ is odd.
One of the following cases arises:

Case 1: $|A|$ is odd. Then A is abelian, and we able to use the following assertion.

Theorem (A. Kondrat'ev, N.M., D. Revin, 2016). *If V is a normal subgroup of a group G and H is a pronormal subgroup of G, then, for any H-invariant subgroup U of V, the equality $U = N_U(H)[H,U]$ holds.*

If $U \leq A$ and U is H-invariant, then (because of A is minimal) either U is trivial or $U = A$.

Thus, $H \, prn \, G$.

Continue reductions

Let G be non-simple, $H \leq G$, A be a minimal non-trivial normal subgroup of G, $G = HA$, and $|G : H|$ is odd.
One of the following cases arises:

Case 2: $|A|$ is even.

Subcase 2.1: A is a 2-group. Then $A \leq H$, a contradiction.

Subcase 2.2: A is a direct product of nonabelian simple groups.

Proposition (W. Guo, N.M., D. Revin, 2018). *Let $S \leq H$ for some $S \in Syl_2(G)$, $T = A \cap S$, $Y = N_A(H \cap A)$, $Z = N_{H \cap A}(T)$, and $^{-} : G \to G/A$ be the natural epimorphism. If $H \cap A \, prn \, A$, then the following statements are equivalent:*
(1) $H \, prn \, G$;
(2) $N_H(T)/Z \, prn \, (N_H(T)N_Y(T))/Z$ *and* $\overline{N_G(H)} = N_{\overline{G}}(\overline{H})$.

Problem

Problem. *Describe direct products of nonabelian simple groups in which the subgroups of odd index are pronormal.*

Conjecture (E. Vdovin and D. Revin, 2012). *The subgroups of odd index are pronormal in all nonabelian simple groups.*

Theorem (A. Kondrat'ev, 2005). *Let G be a finite nonabelian simple group and $S \in Syl_2(G)$. Then $N_G(S) = S$ excluding the following cases:*

(1) $G \cong J_2,\ J_3,\ Suz$ *or* HN;

(2) $G \cong {}^2G_2(3^{2n+1})$ *or* J_1;

(3) G *is a group of Lie type over field of characteristic* 2;

(4) $G \cong PSL_2(q)$ *where* $3 < q \equiv \pm 3 \pmod 8$;

(5) $G \cong PSp_{2n}(q)$, *where* $n \geq 2$ *and* $q \equiv \pm 3 \pmod 8$;

(6) $G \cong PSL_n^{\eta}(q)$, *where* $n \geq 3$, $\eta = \pm$, q *is odd;*

(7) $G \cong E_6^{\eta}(q)$, *where* $\eta = \pm$ *and* q *is odd.*

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(7) $G \cong E_6^{\eta}(q)$, *where* $\eta = \pm$ *and* q *is odd.*

Theorem (A. Kondrat'ev, N.M, D. Revin, 2016). *Let* $G = PSp_{2n}(q)$, *where* $q \equiv \pm 3 \pmod 8$ *and* n *is not of the form* 2^w *or* $2^w(2^{2k}+1)$. *Then* G *has a non-pronormal subgroup of odd index.*

Problem. *Classify nonabelian simple groups in which the subgroups of odd index are pronormal.*

Theorem (A. Kondrat'ev, N.M, D. Revin, 2018). *Let G be a non-abelian simple group, $S \in Syl_2(G)$. If $C_G(S) \leq S$, then exactly one of the following statements holds:*
(1) *The subgroups of odd index are pronormal in G;*
(2) *$G \cong PSp_{2n}(q)$, where $q \equiv \pm 3 \pmod 8$ and n is not of the form 2^w or $2^w(2^{2k}+1)$.*

(A series of 5 papers by W. Guo, A. Kondrat'ev, N.M, and D. Revin from 2015 to 2018.)

Let G be a finite simple group, $H \leq G$, and $|G : H|$ is odd.

Take $S \in Syl_2(G)$ such that $S \leq H$.

Take $g \in N_G(S) \setminus S$ and construct $K = \langle H, H^g \rangle$.

We can assume that $K < G$. Then there exists a maximal subgroup M of G such that $K \leq M$ (here $|G : M|$ is odd).

Do we know M?

General Idea of Proof

Let G be a finite simple group, $H \leq G$, and $|G : H|$ is odd.

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Do we know M?

Classification of primitive permutation representations of odd degree of finite groups (M. Liebeck and J. Saxl (1985) and W. Kantor (1987)).

Complete classification of maximal subgroups of odd index in finite simple groups (N.M. (2009, revised 2018)).

Simple Groups

Pronormality of Subgroups of Odd Index in Finite Simple groups (A. Kondrat'ev, N.M., D. Revin, 2018+, based on CFSG)

Alternating groups, Sporadic groups, and $^2F_4(2)$

Classical Groups

$G = PSL_n(q)$	$G = PSU_n(q)$	$G = PSp_n(q)$	*Orthogonal Groups*
q is even	*q is even*	*q is even*	
q is odd, $C_G(S) \subseteq S$ *for* $S \in Syl_2(G)$	*q is odd,* $C_G(S) \subseteq S$ *for* $S \in Syl_2(G)$	$q \equiv \pm 1 \pmod 8$	
q is odd, $C_G(S) \nsubseteq S$ *for* $S \in Syl_2(G)$	*q is odd,* $C_G(S) \nsubseteq S$ *for* $S \in Syl_2(G)$	$q \equiv \pm 3 \pmod 8$	

Exceptional Groups of Lie type

$E_8(q)$	$E_7(q)$	$E_6(q)$	$^2E_6(q)$	$^3D_4(q)$	$F_4(q)$	$^2F_4(q)$	$G_2(q)$	$^2G_2(q)$	$^2B_2(q)$
		q is even	*q is even*						
		q is odd, $C_G(S) \subseteq S$ *for* $S \in Syl_2(G)$	*q is odd,* $C_G(S) \subseteq S$ *for* $S \in Syl_2(G)$						
		q is odd, $C_G(S) \nsubseteq S$ *for* $S \in Syl_2(G)$	*q is odd,* $C_G(S) \nsubseteq S$ *for* $S \in Syl_2(G)$						

The subgroups of odd index are pronormal
Problem is solved, but the solution depends on some extra conditions
Problem is still open

Direct Products of Simple Groups

Example 1. *Let* $G_1, G_2 \in \{J_1, {}^2G_2(3^{2m+1})\}$. *Then*
(1) *the subgroups of odd index are pronormal in each* G_i;
(2) $G_1 \times G_2$ *contains a non-pronormal subgroup of odd index.*

Example 2. *Let* G_1, G_2 *be finite groups with self-normalized Sylow 2-subgroups. Then Sylow 2-subgroups of the group* $G_1 \times G_2$ *are self-normalized. Therefore, the subgroups of odd index are pronormal in* $G_1 \times G_2$.

Thus, the solution of Problem for all direct product of simple groups is not equivalent to the solution of Problem for all simple groups.

In Progress...

Let G be a finite group such that G contains a non-pronormal subgroup of odd index.

Let $H \leq G$ and $|G : H|$ is odd.

Problem*. *Find a good algorithm to answer the following question: Is H pronormal in G.*

Case $G \cong PSp_{2n}(q)$, where $q \equiv \pm 3 \pmod 8$ and n is not of the form 2^w or $2^w(2^{2k}+1)$ (joint work with Ch. Praeger, in progress).

A survey paper

A.S. Kondrat'ev, N.V. Maslova, D. O. Revin, On the Pronormality of Subgroups of Odd Index in Finite Simple Groups, arXiv:1807.00384 [math.GR]

Thank you for your attention!