On the complexity of testing isomorphism of graphs of bounded eigenvalue multiplicity

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Polynomial-time instances

Question. *For which classes of graphs is* GI *in polynomial time?*

It was in this context that early *group-theoretic* methods have been pioneered, notably, for:

- Graphs of bounded color multiplicity —[Babai 1979, Furst–Hopcroft–Luks 1980]
- Graphs of bounded eigenvalue multiplicity —[Babai–Grigoryev–Mount 1982]
- Graphs of bounded valence —[Luks 1980]

Vertex-colored graphs

A vertex-colored graph $X = (V, E)$ is equipped with a partition of V into subsets of vertices of the same color

$$V = C_1 \,\dot\cup\, \cdots \,\dot\cup\, C_s.$$

- Aut(X) consists of *color-preserving* automorphisms.
- We call $|C_i|$ the *color multiplicity* of a color i.
- If *all* $|C_i| \le k$, then Aut(X) is *embeddable* in the direct product of s symmetric groups

$$S_k \times \cdots \times S_k.$$

Babai's tower-of-groups algorithm

If a graph X's color multiplicity $\leq k$, assuming $\mathrm{Aut}(X)$ is embeddable in $G := S_k \times \cdots \times S_k$, we compute

$$G = G_1 \geq G_2 \geq \cdots \geq G_m \cong \mathrm{Aut}(X)$$

starting from G_1 followed by $G_2, G_3, \ldots$, and finally G_m.

- Each index $|G_i : G_{i+1}| \leq k!$.
- By Lagrange's theorem, $m = O(n \log n)$.
- It thus runs in $O(f(k)n^c)$ time.

—[Babai 1979, Furst–Hopcroft–Luks 1980]

Eigenvalues and automorphisms

Consider a graph $X = (V, E)$ with eigenvalues $\lambda_1, \ldots, \lambda_t$ and their eigenspaces $W_1, \ldots, W_t$ for which:

- $\mathbf{R}^n = W_1 \oplus \cdots \oplus W_t$.
- Each λ_i's multiplicity $d_i = \dim_{\mathbf{R}}(W_i)$.

Consider the *permutation representation* of $\mathrm{Sym}(V)$ on $\mathbf{R}^n$ (which permutes the coördinates of $x \in \mathbf{R}^n$). Then

$$\mathrm{Aut}(X) = \{g \in \mathrm{Sym}(V) : W_i^g = W_i \text{ for } i = 1, \ldots, t\}$$

so that $\mathrm{Aut}(X)$ is *embeddable* in

$$\mathrm{O}(W_1) \times \cdots \times \mathrm{O}(W_t).$$

Partition-decomposition pairs

Given a graph $X = (V, E)$, we partition V into $\mathrm{Aut}(X)$-invariant subsets and decompose $\mathbf{R}^n$ into $\mathrm{Aut}(X)$-invariant subspaces *simultaneously* such that:

$$V = C_1 \dot\cup \cdots \dot\cup C_s \qquad \mathbf{R}^n = W_1 \oplus \cdots \oplus W_t$$

- All vectors in each $p_{W_j}(C_i)$ have the same length.
- Equivalence relations defined by p_{W_j} are "balanced".
- Each $\langle p_{W_j}(C_i) \rangle = 0$ or W_j.

—[Babai–Grigoryev 1982]

Partition-decomposition pairs

Now, $\mathrm{Aut}(X)$ is *embeddable* in:

$$\mathrm{Sym}(C_1) \times \cdots \times \mathrm{Sym}(C_s) \qquad \mathrm{O}(W_1) \times \cdots \times \mathrm{O}(W_t)$$

If X's eigenvalue multiplicity $\leq d$, then $\dim_{\mathbf{R}}(W_j) \leq d$.
In fact, $\mathrm{Aut}(X)$ is *embeddable* in

$$H_1 \times \cdots \times H_s \leq \mathrm{Sym}(C_1) \times \cdots \times \mathrm{Sym}(C_s)$$

such that each $|H_i| \leq n^d$. —[Babai–Grigoryev 1982]

However, little is known about $|C_i|$.

Fixed-parameter tractability

X's color multiplicity $\leq k$: $O(f(k)n^c)$ time
X's eigenvalue multiplicity $\leq d$: $O(n^{f(d)})$ time

In general, if the bounded parameter is *not* involved in the exponent, it is called *fixed-parameter tractable*.

—[Downey–Fellows 1992]

Question. *Under what condition is the problem of bounded eigenvalue multiplicity "reducible" to that of bounded color multiplicity?*

An observation

Let $X = (V, E)$ be a graph of *bounded eigenvalue multiplicity*.

Theorem. *If* $\mathrm{Aut}(X)$ *is primitive, and the rank of* $\mathrm{Aut}(X)$ *is bounded,* X *admits a vertex coloring of bounded color multiplicity.*

- The *rank* of $\mathrm{Aut}(X)$ is the number of orbits of $\mathrm{Aut}(X)$ on its natural action on $V \times V$.
- *Primitive groups of small ranks* are important and well-known.

An observation

We appeal to the following fundamental theorem:

Theorem (Delsarte–Goethals–Seidel 1977). *If S is a finite set of points on the unit sphere in $\mathbf{R}^d$, and $s := |\{(x, y) : x, y \in S \text{ and } x \neq y\}|$, then*

$$|S| \leq \binom{d+s-1}{s} + \binom{d+s-2}{s-1}$$

(where $(,)$ denotes the usual inner product in $\mathbf{R}^d$).

An observation

Let $X = (V, E)$ be a graph such that $\mathrm{Aut}(X)$ acts *primitively* on each of its orbits.

Proposition. *For X's partition-decomposition pair $V = C_1 \dot\cup \cdots \dot\cup C_s$ and $\mathbf{R}^n = W_1 \oplus \cdots \oplus W_t$, let r_i be the rank of $\mathrm{Aut}(X)|_{C_i}$ and $d_j = \dim_{\mathbf{R}}(W_j)$ for $i = 1, \ldots, s$ and $j = 1, \ldots, t$. For each C_i, there is W_j such that*

$$|C_i| \leq \binom{d_j + r_i - 2}{r_i - 1} + \binom{d_j + r_i - 3}{r_i - 2}$$