On the Isomorphism Problem for Cayley combinatorial objects

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Combinatorial Objects

Definition (L. Babai, 1977)

Combinatorial objects are objects of a concrete category, i.e. the category with a forgetful functor to the category of sets.

Definition (P. Pálfy, 1987)

A combinatorial object on a (finite) set Ω is a relational structure, i.e. a (finite) subset of $\Omega \cup \Omega^2 \cup \Omega^3 \dots$.

Definition (N. Brand, 1991)

A finite subset O of $\Omega \cup 2^{\Omega} \cup 2^{2^{\Omega}} \dots$.

In what follows a combinatorial object will mean an ordered tuple $O := (R_1, \dots, R_d)$ where $R_i \subset \Omega \cup \Omega^2 \cup \Omega^3 \dots$.
Isomorphisms and automorphisms are defined in a natural way.

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- Cayley maps,
- etc.

Isomorphism problem for Cayley combinatorial objects

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Given a finite group H and combinatorial objects $O, O' \in \mathrm{O}bj(H_R)$, find whether they are isomorphic and (if so) find an isomorphism between them.

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For any $O \in \mathrm{Obj}(H_R)$ and $f \in \mathrm{Aut}(H)$, the object O^f is a Cayley object over H isomorphic to O. We say that O^f is Cayley isomorphic/equivalent to O. Notation $O \cong_{Cay} O^f$.

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CI-property (Babai, 1977)

A Cayley object O is called a CI-object iff

$$\forall\ O' \in \mathrm{Obj}(H_R)\ \ O' \cong O \iff O' \cong_{Cay} O.$$

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Definition

Let $\mathfrak{K}$ be a class of combinatorial objects. A group H is called a CI-group w.r.t a class $\mathfrak{K}$ ($\mathfrak{K}$-CI-group for short) if any object $O \in \mathfrak{K}$ is a CI-object.

Cayley digraphs

Definition

Let $S \subseteq H$ be a subset of a finite group H. A Cayley digraph $\mathrm{Cay}(H, S)$ has H as a vertex set; two vertices $x, y \in H$ are connected iff $xy^{-1} \in S$. If $S = S^{(-1)}$ and $1_H \notin S$, then $\mathrm{Cay}(H, S)$ is a simple undirected graph.

A colored Cayley digraph is a tuple $(\mathrm{Cay}(H, S_0), \ldots, \mathrm{Cay}(H, S_d))$ where $S_0, \ldots, S_d$ are pairwise disjoint non-empty subsets of H. Notation, $\mathrm{Cay}(H, \mathcal{S}), \mathcal{S} = (S_0, \ldots, S_d)$.

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Isomorphism between colored Cayley digraphs

Two colored Cayley digraphs $\mathrm{Cay}(H, (S_0, \ldots, S_d))$ and $\mathrm{Cay}(H, (S_0', \ldots, S_d'))$ are isomorphic iff there exists $g \in \mathrm{Sym}(H)$ s.t. $\mathrm{Cay}(H, S_i)^g = \mathrm{Cay}(H, S_i'), i = 0, \ldots, d$. If $g \in \mathrm{Aut}(H)$, then the digraphs are called Cayley isomorphic.

IP for Cayley digraphs

IP for Cayley graphs v. 1.0

Given a group H of order n and $S, T \subseteq H$, decide whether $\mathrm{Cay}(H, S)$ and $\mathrm{Cay}(H, T)$ are isomorphic.

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Given two groups H, K of order n and $S \subseteq H$, $T \subseteq K$, decide whether $\mathrm{Cay}(H, S)$ and $\mathrm{Cay}(K, T)$ are isomorphic.

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Proposition

If there exists an algorithm that solves version 1.0 for all groups in a time $f(n)$, then

1. it solves version 2.0 in $f(n^2)$;
2. it solves the Group Isomorphism Problem in a time $f(n^4)$

Isomorphism problem for circulant graphs

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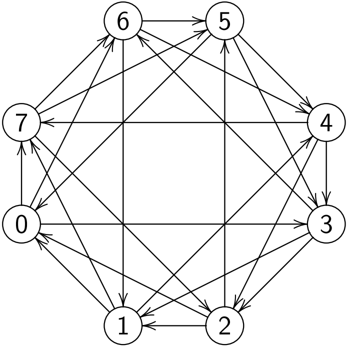
Minimal Counterexample (Elspas and Turner, 1970).



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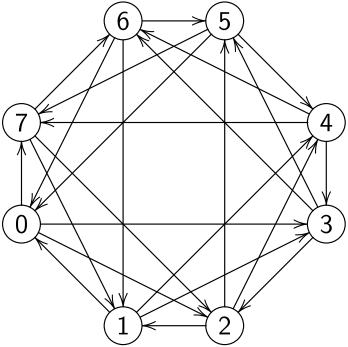

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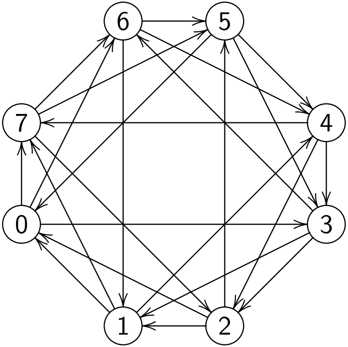

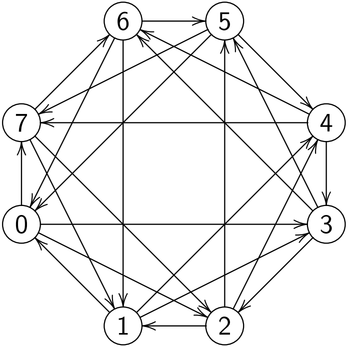

$$\mathrm{Cay}(\mathbb{Z}_8, \{1, 2, 5\}) \xrightarrow{(2,6)(3,7)} \mathrm{Cay}(\mathbb{Z}_8, \{1, 6, 5\})$$

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Theorem (Alspach & Parsons and Egorov & Markov, 1979)

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Ádám's conjecture is true if

1. n is a prime - Elspas & Parsons;
2. $n = 2p, 3p, 4p$ - Babai 1977;
3. $n = pq, p \neq q$ are primes - C. Godsil (1977), Klin & Pöschel (1978), Alspach & Parsons (1979)

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Pálfy's correction of Ádám's conjecture (1987):

Ádám conjecture is true if n is a square free or twice square free number.

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Theorem (M. 1995-97)

The corrected Ádám's conjecture is true.

CI-groups w.r.t. digraphs ($\vec{\mathfrak{G}}$-groups)

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During last 50 years the classification of CI-groups w.r.t. digraphs was studied by many researches: B. Alspach, L.Babai, M.Conder, E. Dobson, B. Elspas, V.N.Egorov, P.Frankl, C. Godsil, M. Hirasaka, Y.-Q. Feng, M. Klin, I. Kovacs, C.H.Li, Z.P. Lu, A.I.Markov, L. Nowitz, T.D. Parsons, P. Pálfy, R. Pöschel, C. Praeger, P. Spiga, G. Somlai, J. Turner.

Theorem (necessary conditions to be a CI-group w.r.t. digraphs)

If H is a CI-group w.r.t. digraphs, then H is a coprime product of groups from the following list:

$$\mathbb{Z}_p^e, \quad \mathbb{Z}_4, Q_8, A_4, E(M,2), E(M,4).$$

where M is a direct product of elementary abelian groups of odd order.

CI-groups w.r.t. digraphs (sufficient conditions)

Theorem

The following groups are CI-groups w.r.t. digraphs

1. $\mathbb{Z}_n$ where n is square-free or twice square-free number;
2. $\mathbb{Z}_p^e, e \leq 5$;
3. $\mathbb{Z}_p^2 \times \mathbb{Z}_q, \mathbb{Z}_p^3 \times \mathbb{Z}_q$ where p and q are distinct primes;
4. $D_{2p}, \mathbb{Z}_p \rtimes \mathbb{Z}_4$;
5. $Q_8, Q_8 \times \mathbb{Z}_p, A_4$;
6. $D_{2n}, \mathbb{Z}_p^2 \times \mathbb{Z}_n, \mathbb{Z}_p^2 \times \mathbb{Z}_q \times \mathbb{Z}_n$ with $\gcd(n, \varphi(n)) = 1$

1. C.H. Li, On isomorphisms of finite Cayley graphs - survey, DM 256 (2002),
2. C.H. Li, Z.P. Lu, P. Pálfy, Further restrictions on the struture of finite CI-groups, JACO 26 (2007).

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3. $n = 2^m$, Muzychuk & Pöschel, 1999;
4. arbitrary n, Evdokimov & Ponomarenko - 2003, M.-2004.

An isomorphism problem for arbitrary cyclic combinatorial objects of orders p^2 and pq was solved by Job, Huffman and Pless in 1993,1996.

Solving sets

Definition

A subset $P \subset \mathrm{Sym}(H)$ is called a solving set for a Cayley digraph $\mathrm{Cay}(H, S)$ iff

$$\forall_{T \subseteq H} \mathrm{Cay}(H, S) \cong \mathrm{Cay}(H, T) \iff$$

$$\iff \exists_{p \in P} \mathrm{Cay}(H, S)^p = \mathrm{Cay}(H, T).$$

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A Cayley object $O \in \mathrm{Obj}(H_R)$ is CI iff any regular subgroup of $\mathrm{Aut}(O)$ isomorphic to H is conjugate to H_R in $\mathrm{Aut}(O)$.

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Let S be an arbitrary subset of H. Let $F_i, i \in I$ be an H-base of the group $G := \mathrm{Aut}(\mathrm{Cay}(H,S)))$. Denote by $f_i \in \mathrm{Sym}(H)$ permutations such that $H_R = F_i^{f_i}, i \in I$. Then $\cup_{i\in I} f_i \,\mathrm{Aut}(H)$ is a solving set for $\mathrm{Cay}(H,S)$.

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5. $\langle \rho \rangle = \langle \sigma \rangle^{(2,6)(3,7)} \implies \mathrm{Aut}(\mathbb{Z}_8) \cup (2, 6)(3, 7)\,\mathrm{Aut}(\mathbb{Z}_8)$ is a solving set for $\mathrm{Cay}(\mathbb{Z}_8, \{1, 2, 5\})$.

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4. Klin-Pöschel approach - use the method of Schur rings to find all possible automorphism groups.

Coherent closure of a Cayley graph

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These special partitions are called S-partitions of H. They are in 1-1 correspondence with Schur rings over H.

Schur rings (algebras)

Definition (Wielandt)

Let $S \subseteq H$. An element $\underline{S} := \sum_{s \in S} s \in \mathbb{Q}[H]$ is called a simple quantity. We abbreviate $\underline{\{g\}}$ as $\underline{g}$.

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A subalgebra $\mathcal{A}$ of $\mathbb{Q}[H]$ arising in this way is called a Schur algebra/ring.

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   2. Find $f_i \in \mathrm{Sym}(H)$ with $H_R = F_i^{f_i}, i = 1, \ldots, k$;
   3. Set $P(\mathcal{S}) := \bigcup_{i=1}^{k} f_i \,\mathrm{Aut}(H)$;
3. Take $\bigcup_{\mathcal{S} \in \mathfrak{S}} P(\mathcal{S})$ as a solving set for Cayley digraphs over H.

This scheme successfully worked for $\mathbb{Z}_n$ if n is a power of an odd prime or a product of two distinct primes.

Example: solving set for circulant graphs of order 8

The following list was generated by the computer program COCO (thanks to Misha Klin).

$\{0\}$, $\{1, 2, 3, 4, 5, 6, 7\}$;
$\{0\}$, $\{1, 3, 5, 7\}$, $\{2, 6, 4\}$;
$\{0\}$, $\{1, 3, 5, 7, 2, 6\}$, $\{4\}$;
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Example

N	S-partition $\mathcal{S}$	Aut. group	cyclic bases	Solving set
1	$\{0\}, \{1,2,3,4,5,6,7\}$	S_8	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
2	$\{0\}, \{1,3,5,7\}, \{2,6,4\}$	$S_2 \wr S_4$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
3	$\{0\}, \{1,3,5,7,2,6\}, \{4\}$	$S_4 \wr S_2$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
4	$\{0\}, \{1,3,5,7\}, \{2,6\}, \{4\}$	$S_2 \wr S_2 \wr S_2$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
5	$\{0\}, \{1,3,5,7\}, \{2\}, \{6\}, \{4\}$	$S_2 \wr \mathbb{Z}_4$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
6	$\{0\}, \{1,5\}, \{3,7\}, \{2\}, \{6\}, \{4\}$	$\mathbb{Z}_8.Z_2$	$\langle\rho\rangle, \langle\sigma\rangle$	$\mathbb{Z}_8^* \cup \alpha\mathbb{Z}_8^*$
7	$\{0\}, \{1,5\}, \{3,7\}, \{2,6\}, \{4\}$	$\mathbb{Z}_4 \wr S_2$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
8	$\{0\}, \{1,3\}, \{5,7\}, \{2,6\}, \{4\}$	$\mathbb{Z}_8.\mathbb{Z}_2$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
9	$\{0\}, \{1,7\}, \{3,5\}, \{2,6\}, \{4\}$	D_{16}	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
10	$\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}$	$\mathbb{Z}_8$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$

Here $\alpha = (2,6)(3,7)$. Thus $\mathbb{Z}_8^* \cup \alpha\mathbb{Z}_8^*$ is a solving set for circulant graphs over $\mathbb{Z}_8$.

Solution of the isomorphism problem for circulant digraphs.

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Theorem (Klin-Pöschel, 1978)

Let n be an odd prime power. Then

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Theorem (Muzychuk-Pöschel, 1999)

Let $n = 2^m$. Then there exists an efficiently constructed solving set P_n for colored circulant digraphs of order n s.t. $|P_n| \leq n^C \varphi(n)$.

But if n is a square-free number, the number of S-partitions is not polynomial in n.

Control of H-bases

Definition

Let $H_R \le X \le Y \le \operatorname{Sym} H$ be arbitrary subgroups. We say that X controls H-bases of Y, notation $X \preceq_H Y$, if any H-base of X contains an H-base of Y.

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The following are equivalent

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The relation $\preceq_H$ is a partial order on the lattice $[H_R, \operatorname{Sym}(H)]$.

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Example. The symmetric group Sym(8) has two $\prec_{\mathbb{Z}_8}$-minimal subgroups: $\mathbb{Z}_8$ and $\mathbb{Z}_8 \rtimes \langle \sigma \rangle$ where $\sigma(x) = 5x$.

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Theorem (Pálfy, 1987)

If H is a cyclic group of order n, then H_R is a unique $\preceq_H$-minimal subgroup iff $n = 4$ or $\gcd(n, \varphi(n)) = 1$.

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Theorem (M., 1999)

If H is cyclic, then each $\prec_H$-minimal subgroup of $X \in [H_R, \operatorname{Sym} H]$ is solvable and $\pi(X) = \pi(H)$.

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Theorem (M., 2004)

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Theorem (M. 2004)

Let $n = p_1^{m_1} \cdot \ldots \cdot p_k^{m_k}$ be a decomposition of n into a product of prime powers. Denote by $P_{p_i^{m_i}}$ a solving set for colored circulant digraphs over $\mathbb{Z}_{p_i^{m_i}}$.Then the set $P_n := P_{p_1^{m_1}} \times \ldots \times P_{p_k^{m_k}}$ is a solving set for colored Cayley digraphs over $\mathbb{Z}_n$. In particular, $|P_n| < n^C \varphi(n)$.

Isomorphism problem for cyclic combinatorial objects

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Let $n = p_1 \cdots p_k$ be a prime decomposition of $n, p_1 \geq ... \geq p_k$.
Define a subgroup W_n inductively:

$$W_n := \begin{cases} AGL_1(p_k) & \text{if } k = 1; \\ AGL_1(p_k) \wr W_{n/p_k} & \text{if } k > 1. \end{cases}$$

The action of $AGL_1(p_k) \wr W_{n/p_k}$ on $\overline{n}$ is defined via the bijection:

$$\overline{n} \ni i \leftrightarrow (q, r) \in \overline{p_k} \times \overline{n/p_k}, \text{ where } i = q\frac{n}{p_k} + r, 0 \leq r < n/p_k.$$

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Theorem (M. & Ponomarenko, 2017)

The group W_n is a solving set for all C-invariant combinatorial objects. In other words, $O, O' \in \mathrm{O}bj(C)$ are isomorphic iff there exists an element $f \in W_n$ s.t. $O^f = O'$.

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Unfortunately, $|W_n|$ is not polynomial in n. But the group W_n is solvable. This yields the following result.

Theorem (M. & Ponomarenko 2017)

The isomorphism of any two cyclic objects can be tested in time polynomial in their sizes.

Non-graphical cyclic combinatorial objects

Theorem (M., 2011)

The set P_n is also a solving set for a semisimple cyclic codes of length n. In other words, two semisimple cyclic codes $C, D \leq \mathbb{F}_q^n$ are permutation equivalent iff there exists $g \in P_n$ s.t. $C^g = D$.

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Theorem (I. Kovacs, D. Marušič and M. Muzychuk, 2015)

A cyclic group is a CI-group with respect to balanced/symmetric configurations.

Non-graphical combinatorial objects: Cayley maps

Definition

A Cayley map is a triple $M(H, S, \rho)$ where

1. H is a finite group;
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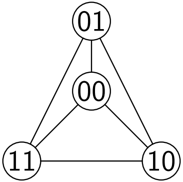

Map isomorphisms

Definition

Two Cayley maps $M(H, S, \rho)$ and $M(H, S', \rho')$ are isomorphic iff there exists a bijection $f \in \mathrm{Sym}(H)$ s.t.

$$\{(h, sh, \rho(s)h) \mid s \in S, h \in H\}^f = \{(h, sh, \rho'(s)h) \mid s \in S', h \in H\}.$$

Cayley isomorphism

Two Cayley maps $M(H, S, \rho)$ and $M(H, S', \rho')$ are Cayley isomorphic iff there exists $f \in \mathrm{Aut}(H)$ s.t. $S^f = S$ and $f_S \rho = \rho' f_S$.

Problem

Classify all finite groups with CI-property with respect to maps.

CI-groups with respect to maps

Theorem (M and G. Somlai, 2015)

Let H be a CI-group with respect to Cayley maps. Then H is isomorphic to one of the following groups

1. $\mathbb{Z}_2^r \times \mathbb{Z}_m, \mathbb{Z}_4 \times \mathbb{Z}_m, \mathbb{Z}_8 \times \mathbb{Z}_m, Q_8 \times \mathbb{Z}_m$;
2. $\mathbb{Z}_m \rtimes \mathbb{Z}_{2^e}, e = 1, 2, 3.$

where m is a square-free odd number.

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