Diagonal limits of cubes over a finite alphabet

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Hamming space

Let $B = \{b_1, \dots, b_q\}$ be an alphabet, $q \geq 2$. Denote by $H_n(q)$ the *Hamming space* of dimension n on alphabet B. This space consists of all n-tuples $(a_1, \dots, a_n)$, $a_i \in B$, $1 \leq i \leq n$, where the distance d_{H_n} between two n-tuples is equal to the number of coordinates where they differ.
The *scaled Hamming space* $\hat{H}_n(q)$ have the same set of points, but the distance is defined as $\frac{1}{n} d_{H_n}$.

Divisible sequence

A sequence of positive integers $\tau = (m_1, m_2, \ldots)$ is called *divisible* if $m_i | m_{i+1}$ for all $i \in \mathbb{N}$.
Denote by $(s_1, s_2, \ldots)$ the sequence of ratios of the sequence τ, i.e.

$$s_1 = m_1, \qquad s_{i+1} = \frac{m_{i+1}}{m_i},\ i \geq 1.$$

For any $i \geq 1$ define an isometric embedding $\psi_{s_i} : \hat{H}_{m_i}(q) \to \hat{H}_{m_{i+1}}(q)$ by the rule:

$$\psi_{s_i}(x_1, \ldots, x_{m_i}) = (\underbrace{x_1, \ldots, x_{m_i} | x_1, \ldots, x_{m_i} | \ldots | x_1, \ldots, x_{m_i}}_{s_i \cdot m_i}).$$

Then the sequence τ determines the directed system of scaled Hamming spaces on the alphabet B

$$\langle \hat{H}_{m_i}(q), \psi_{s_i} \rangle_{i \in \mathbb{N}}$$

with the diagonal embeddings ψ_{s_i}, $i \geq 1$.

The limit space of the directed system

$$\mathcal{H}(\tau, q) = \varinjlim \langle \hat{H}_{m_i}(q), \psi_{s_i} \rangle$$

is called a *diagonal limit* of the sequence of spaces $\hat{H}_{m_i}$, $i \geq 1$. We call the metric space $(\mathcal{H}(\tau, q))$ the τ*-periodic Hamming space* over the alphabet B.

Limit of Hamming spaces

In [1] Peter J. Cameron and Sam Tarzi considered $H(\tau, 2)$ over alphabet $\{0, 1\}$.

A. $H(\tau, 2)$ isometric to the space of finite unions of half-open subintervals of the interval $[0, 1)$ with some special rational endpoints;
B. Completions of $H(\tau, 2)$ independent of choice of τ;
C. For any τ the space $H(\tau, 2)$ is homogeneous.

Limit of Hamming spaces

In [2] periodic Hamming spaces $H(\tau, 2)$ over alphabet $\{0, 1\}$ were regarded as spaces of clopen subsets of boundaries of spherically homogeneous rooted trees.

Questions

In [1] P. J. Cameron and S. Tarzi formulated the questions:

(A) Is there an other representation of $H(\tau, q)$, $q > 2$?

(B) Are completions of diagonal limits Hamming spaces $\mathcal{H}(\tau, q)$ on alphabet B independent of choice of τ?

Rooted tree

Let T be an infinite locally finite spherically homogeneous rooted tree with the root v_0, n be some nonnegative integer.
A spherically homogeneous rooted tree T is uniquely defined by its *spherical index*, i.e. by an infinite sequence of positive integers $[s_1; s_2; \ldots,]$ such that s_i is the number of edges joining a vertex of the $(i-1)$th level with vertices of the ith level, $i \geq 1$.
If the tree (T, v_0) has spherical index $[s_1; s_2; \ldots,]$ then the sequence $|L_i| = m_i$, $i \geq 1$, is divisible.

Boundary of tree

The boundary ∂T of a tree T is the set of infinite rooted paths. Define a distance ρ on the set ∂T as

$$\rho(\gamma_1, \gamma_2) = \begin{cases} \frac{1}{k+1}, & \text{if } \gamma_1 \neq \gamma_2 \\ 0, & \text{if } \gamma_1 = \gamma_2 \end{cases},$$

where k is the length of the common beginning of rooted paths γ_1 and γ_2.

Bernoulli measure

Define the Bernoulli measure μ on the Borel σ-algebra of ∂T_τ by the rule:

$$\mu(C_v) = \frac{1}{n_v},$$

where n_v is the number of vertices of T_τ on the level containing the vertex v, C_v is cylindrical set corresponding to v (see [3]).

Introduce the discrete metric ϱ on the alphabet B, i.e.

$$\varrho(b_i, b_j) = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \end{cases},$$

for all $1 \leq i, j \leq q$. The metric ϱ induces the discrete topology on B. Denote by $C(\partial T_\tau, B)$ the set of all continuous functions from the space ∂T_τ to the space B.

Define a metric $d_\mu : C(\partial T_\tau, B) \times C(\partial T_\tau, B) \to \mathbb{R}^+$ by putting

$$d_\mu(f, g) = \mu(\Sigma_f \triangle \Sigma_g),$$

where $f, g \in C(\partial T_\tau, B)$ and the symmetric difference of $\Sigma_f = \{f^{-1}(b_i), 1 \leq i \leq q\}$ and $\Sigma_g = \{g^{-1}(b_i), 1 \leq i \leq q\}$ defined by the rule:

$$\Sigma_f \triangle \Sigma_g = \bigcup_{i \neq j} (f^{-1}(b_i) \cap g^{-1}(b_j)).$$

Theorem [4]

The τ-periodic Hamming space $\mathcal{H}(\tau, q)$ on the alphabet B is isometric to the space of all continuous functions $C(\partial T_\tau, B)$ with the metric d_μ.

This theorem is the answer to the question (A).

The following statement is the answer to the question (B).

Corollary [4]

For every infinite strictly increasing divisible sequence $\tau = (m_1, m_2, \ldots)$ the completion of the space $\mathcal{H}(\tau, q)$ is isometric to the completion of the space $\mathcal{H}(\beta, q)$, where $\beta = (2, 4, 8, \ldots)$.

References

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Thank you for your attention!