Locally toroidal abstract polytopes and Y-presentations of sporadic groups

(i) Coxeter groups and C-string groups

(ii) Y-shaped presentations for the Monster and its subgroups

(iii) examples of (i) from (ii)
— old and new

(iv) conclusion and open problems

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Coxeter groups

A Coxeter group $G = \langle g_1, g_2, \dots g_n \mid \mathcal{F} \rangle$ of rank n is generated by n involutions $g_1, \dots, g_n$, subject to relations $\mathcal{F}$

$$\mathcal{F} = \{ g_1^2, g_2^2, \dots, g_n^2, (g_1 g_2)^{e_{12}}, \dots (g_i g_j)^{e_{ij}}, \dots (g_{n-1} g_n)^{e_{n-1,n}} \}$$

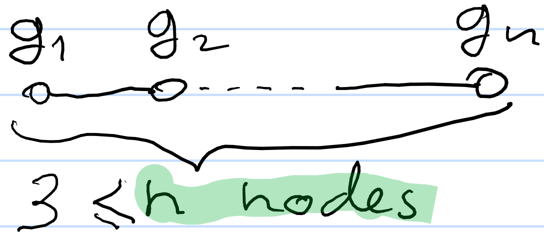
$$e_{ij} \in \{2, 3, \dots\} \cup \{\infty\}.$$

Usually drawn as a Coxeter diagram a graph on n nodes, node i is joined to node j by $e_{ij} - 2$ edges. E.g. $G = S_n$:

Classification of finite irreducible Coxeter groups

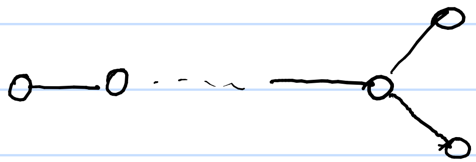
Irreducible = with connected diagram (connected components = commuting normal subgroups).

$A_n \cong S_{n+1}$



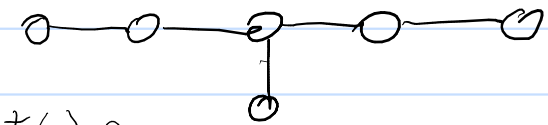
$g_k = (k, k+1)$ - transposition

$D_n \cong 2^{n-1}:S_n$

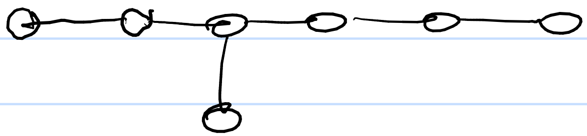


$D_3 = A$

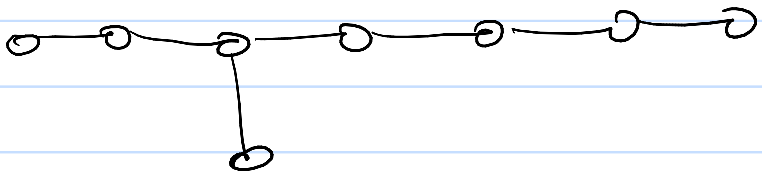
$E_6 \cong U_4(2).2$:



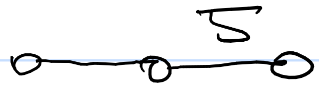
, $E_7 \cong 2 \times Sp$



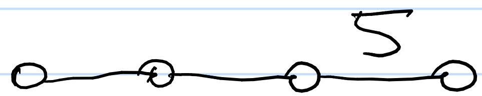
$E_8 \cong 2.O_8^+(2).2$:



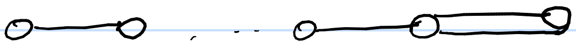
H_3 :



H_4 :



B_n :



$(\cong 2^n:S_n)$

F_4 :

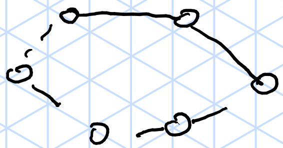
(order 1152)

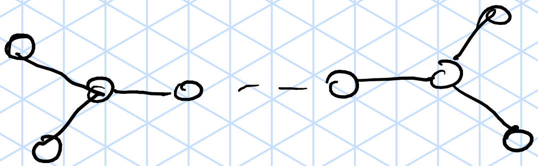
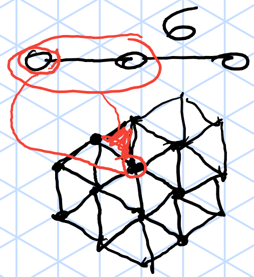
Affine Coxeter groups and honeycombs

Finite Coxeter groups are restricting as Lego blocks

Idea — take infinite examples, and their finite quotients.

Affine Coxeter grps — finite ones, acting on lattices.

$\tilde{A}_n$:  $n+1$ nodes, $\tilde{A}_n \cong \mathbb{Z}^n : A_n$

$\tilde{D}_n$: , $\tilde{G}_2$: 

$= \mathbb{Z}^2 : D_{12}$

Honeycomb

$\tilde{B}_n$, $\tilde{E}_n$ . . .

$\tilde{F}_4$: , $\tilde{F}_4 \cong \mathbb{Z}_4 : F_4$

Honeycomb — tiling of $\mathbb{R}^n$ induced by embedding $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$.

C-string groups and abstract regular polytopes

Finite irreducible Coxeter groups – regular polytopes

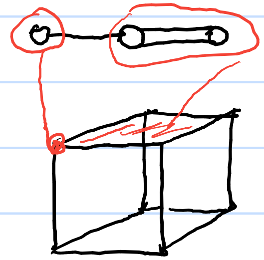
A_n – simplices

B_n – hypercubes (and dual)

F_4 – 24-cell

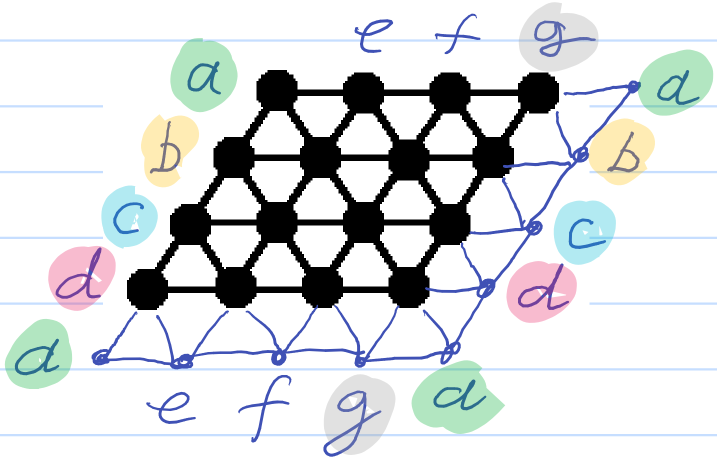
etc.

B_3



Finite quotients of affine Coxeter groups give toroids. – examples of abstract regular polytopes

E.g. Shrikhande graph (a 16-vertex non-Schurian s.r.g.)



← Shrikhande graph on the torus (note identification)

Parabolics, C-string groups, intersection property

$G := \langle g_1, \dots, g_n \mid \mathcal{I} \rangle$ – Coxeter group (a quotient of)

$G \geq S_I := \langle g_{i_1}, \dots, g_{i_k} \rangle$ – parabolic subgroup

$I = \{i_1, \dots, i_k\}$

G satisfies Intersection Property (IP) if

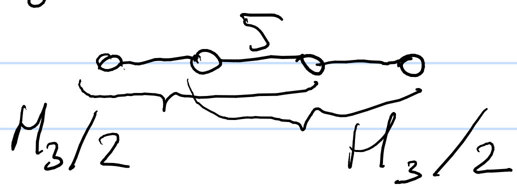
$\forall\ S_I, S_J :\ S_I \cap S_J = S_{I \cap J}$

C-string group: a (quotient of) Coxeter group, with (IP), and string diagram.

Abstract regular polytope: the incidence system of cosets of parabolics in a C-string group.

B. Grünbaum, 197?
H.M.S. Coxeter
J. Tits
E. Schulte 1980
P. McMullen 198?

e.g. 11-cell $\quad H_3 \cong 2 \times A_5$



Universal, locally X,... abstract regular polytopes

All relations are in proper parabolics
$\Longleftrightarrow$ G is universal

E.g. 11-cell is universal, finite Coxeter groups are universal, etc.

Closely related to thin diagram geometries (Tits-Buekenhout) with string diagram and their universal covers.

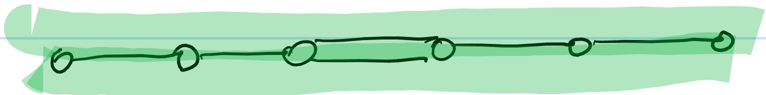
Extremely useful to study finitely presented groups. (need not restrict to string diagrams, naturally).
Types of parabolics fixed to X - talk about locally X object (11-cell is locally icosahedral).

Locally toroid a.r. polytopes

* Parabolics either spherical (= finite Coxeter groups) or toroid (= finite quotients of affine c.g.)

Thm. (199?, McMullen-Schulte-...)
rank of proper locally toroid polytope is at most 6.

Today we talk about one of rank 6 cases:



In 1996 McMullen and Schulte published a list of universal finite examples, and conjectured its completeness. We give one more example (and remove a non-existent series of examples).

The known universal finite polytopes $\{\{3,3,4,3\}_{\mathbf{s}}, \{3,4,3,3\}_{\mathbf{t}}\}$, their groups, and (whenever they exist) "Y-overgroups".

s	t	v	f	G	"Y-overgroup"
(2000)	(2000)	2^5	2^5	$[2^{15}3^2]$	$O_8^-(2){:}2$
(2000)	(2200)	2^5	2^7	$[2^{18}3^2]$	$2^8{:}O_8^-(2){:}2$
(2200)	(2200)	2^{11}	2^{11}	$[2^{24}3^2]$	–
(3000)	(3000)	$2^2{\cdot}3^2{\cdot}5{\cdot}13$	$2^2{\cdot}3^2{\cdot}5{\cdot}13$	$(3^2 \times L_4(3)).2^2$	–
(2200)	(3000)	$2^9{\cdot}3{\cdot}5^2{\cdot}7$	$2^3{\cdot}3^5{\cdot}5^2{\cdot}7$	$S_3 \times O_8^+(2){:}S_3$	Fi_{22}

New example

s, t : F_4 acts on $\mathbb{Z}^4 \cup \left(\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \mathbb{Z}^4\right)$ in $\tilde{F}_4$, and invariant sublattices are cyclic, gen. by either $(y,0,0,0)$ or by $(y,y,0,0)$, with $y \in 2\mathbb{Z}^+$.

New example is "mixed characteristic"! Modular representation is not helpful...

Y-presentations - enter sporadic simple groups

Group	Structure	Relations	Centre
Y_{mn0}	S_{m+n+2}	none	
Y_{111}	2^3S_4	none	$(ab_1b_2b_3)^3$
Y_{211}	2^4S_5	none	
Y_{311}	2^5S_6	none	$(ab_1b_2b_3c_1d_1)^5$
Y_{411}	2^6S_7	none	
Y_{511}	2^6S_8	$P=1$	
Y_{221}	$O_6^-(2):2$	none	
Y_{321}	$O_7(2) \times 2$	none	f_{12}
Y_{421}	$O_8^+(2):2$	$Q=1$	
Y_{331}	$2^2.2^6.O_7(2)$	$R=1$	f_{12}, f_{21}
Y_{431}	$O_9(2) \times 2$	$R=1$	f_{21}
Y_{441}	$O_{10}^-(2):2$	$R=1$	
Y_{222}	$3^5:O_5(3):2$	$S=1$	
Y_{322}	$O_7(3) \times 2$	$S=1$	f_{12}
Y_{422}	$O_8^+(3):2$	$Q=S=1$	
Y_{332}	$2^2.Fi_{22}$	$S=1$	f_{12}, f_{21}
Y_{432}	$2 \times Fi_{23}$	$S=1$	f_{21}
Y_{442}	$3Fi_{24}$	$S=1$?	
Y_{333}	$2^3.{}^2E_6(2)$	$S=1$?	f_{12}, f_{23}, f_{31}
Y_{433}	$2^2.B$	$S=1$?	f_{21}, f_{31}
Y_{443}	$2 \times M$	$S=1$?	f_{31}
Y_{444}	$M \text{ wr } 2$	$S=1$?	

can be removed
(A. A. Ivanov, S. Norton)

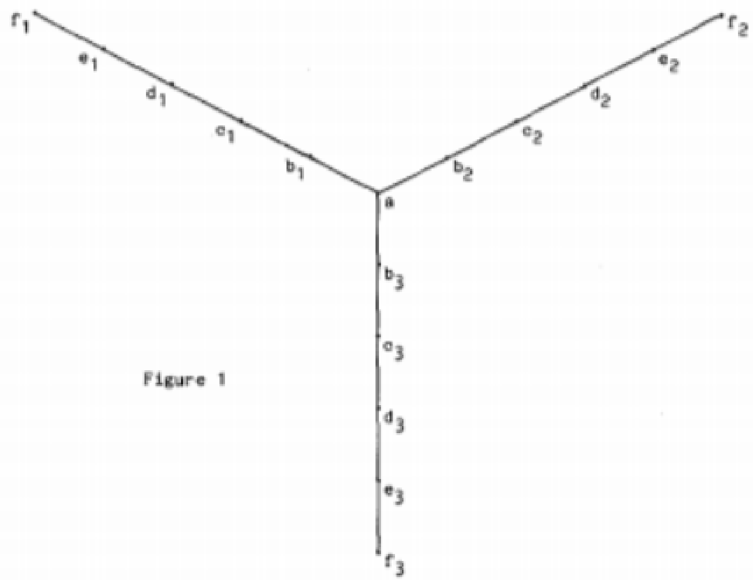


Figure 1

$P = (ab_1b_2b_3c_1d_1e_1f_1)^7$

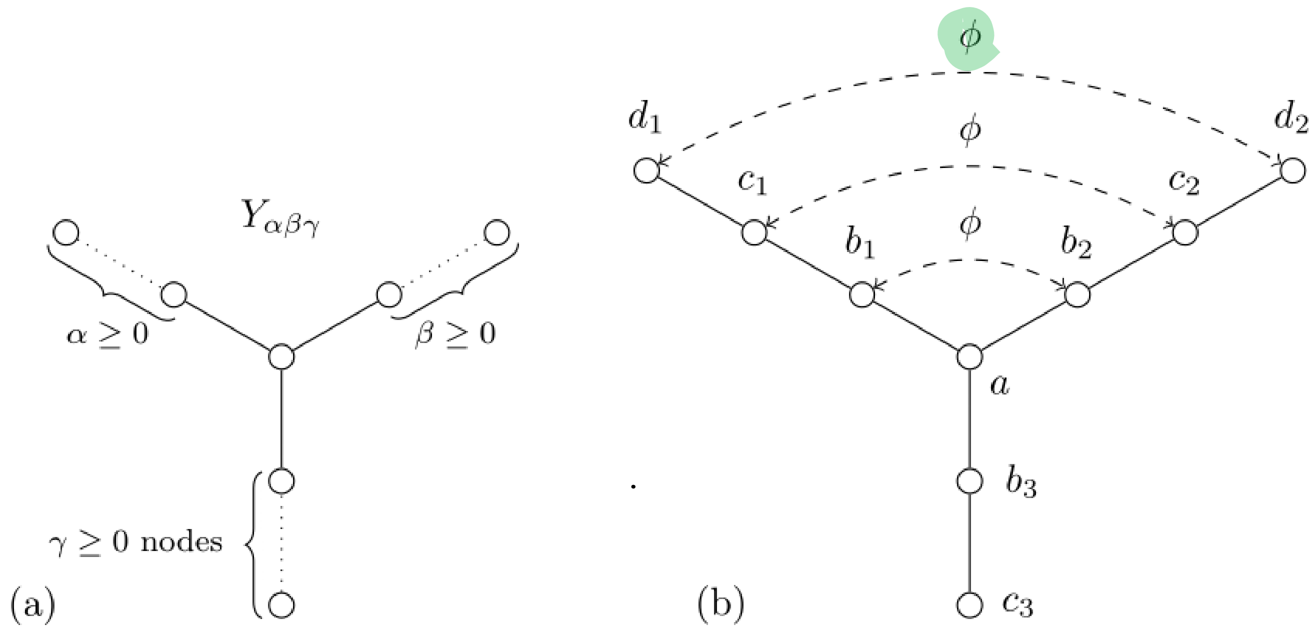
$Q = (f_{12}e_1)^3$ or $(ab_1b_2b_3c_1c_2d_1e_1)^{15}$

$R = [f_{12}, d_2]$ or $[f_{21}, d_1]$

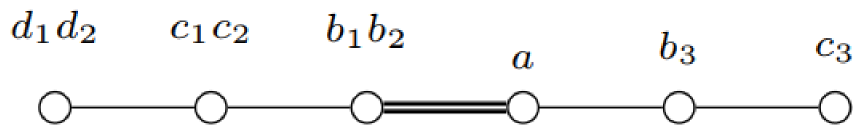
$S = (ab_1c_1ab_2c_2ab_3c_3)^{10}$

From *Atlas of Finite Groups* (1985) by J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson.

Twisting Y-diagrams



Twisting $Y_{332} \cong Fi_{22}$



$Y_{\alpha\beta\gamma}$ has extra relations $1 = S = f_{ij}$

$S = (ab_1c_1ab_2c_2ab_3c_3)^{10}$, $f_{ij} = (ab_ib_jb_kc_ic_jd_i)^9$, where $\{i, j, k\} = \{1, 2, 3\}$

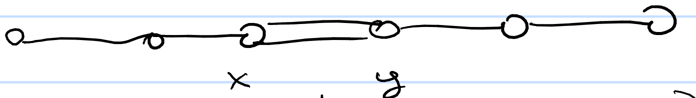
subgroups that S, f_{ij} generate stay invariant under ϕ

Theorem 1. *Let Γ be a universal locally toroid rank 6 abstract regular polytope with vertex figures of type $\{3,4,3,3\}_{(3000)}$ and facets of type $\{3,3,4,3\}_{(2200)}$. Then Γ is the 24-fold cover of Γ_1, where Γ_1 has $v = 11200 = 2^6{\cdot}5^2{\cdot}7$ vertices and $f = 14175 = 3^4{\cdot}5^2{\cdot}7$ facets, and the group $\Omega := O_8^+(2){:}S_3$, of order $g = 2^{13}{\cdot}3^6{\cdot}5^2{\cdot}7$. The group of Γ is isomorphic to $S_4 \times \Omega$.*

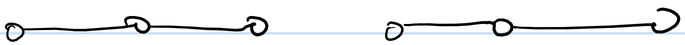
Γ_k is $k!$-fold cover of Γ_1, and

$\mathrm{Aut}(\Gamma_k) \cong S_k \times \Omega$.

These arise from homomorphism ψ



$\downarrow \;\leftarrow\; (xy)^2 \in \mathrm{Ker}\,\psi$



(only one copy of $S_4 \cong A_3$ survives due to extra relations)

Sketch of proof

Universality – computer (coset enumeration ☹)
Big index, would be hard 20 years ago...

Existense (1): Γ_1 exists in Fi_{22}, by twisting Y_{332}, covers exist by embedding our f.p. group into $S_4 \times \Omega$.

Existence (2): Γ_1 exists by taking a permutation representation of F_4 on 24 points, and computing two extra matrices to obtain $\Omega \hookrightarrow GL_{24}(\mathbb{F}_2)$. (These matries lie in the coherent configuration of F_4, this helps.)

$\sigma := d^{cb}$, $\tau := c^{de}$, $1 = (a\sigma\tau)^4 = (f\tau\sigma\tau)^3$

The presentation for $\mathrm{Aut}(\Gamma)$.

$$\hat{b} = \prod_{i\in\{0,8,16\}} (i+3, i+8)(i+4, i+7), \qquad \hat{c} = (3,6)(4,5)(11,14)(12,13)(17,24)(18,23),$$

$$\hat{d} = (5,6)\prod_{i=9}^{16}(i, i+8), \qquad \hat{e} = (21,22)\prod_{i=1}^{8}(i, i+8).$$

F_4 acting on 24 points

$$A := \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&1&1&1&1&0&0\\ 1&0&1&0&0&0&0&1\\ 1&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&1\\ 1&0&0&0&0&1&0&1\\ 0&0&1&1&1&1&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \qquad F := \begin{pmatrix} 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0 \end{pmatrix}$$

$$\hat{a} := I_3 \otimes A, \qquad \hat{f} := \begin{pmatrix} I_8 + E_{21} & 0 & 0\\ 0 & 0 & F\\ 0 & F & 0 \end{pmatrix}.$$

Extra matrices to get $\Omega \hookrightarrow GL_{24}(\mathbb{F}_2)$.

Further extensions (W. I. P.)

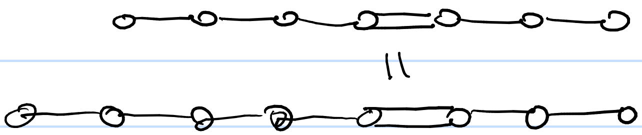
Twisting $Y_{442} \cong 3Fi_{24}$

ACE

Extending $\Gamma_3 \cong S_3 \times \Omega$ gives $(3 \times 2.Fi_{22}).2$ (computer)

index $741312 =$

12 times index of Ω in Fi_{22},



cannot be extended → akin to $Y_{5pq} = Y_{4pq}$ (for p, q big enough)

The Known Finite Polytopes $\{\{3, 3, 3, 4\}, \{3, 3, 4, 3\}_{s}\}$.

s	v	f	g
(2, 0, 0, 0)	20	960	3 68640
(2, 2, 0, 0)	160	30720	117 96480
(3, 0, 0, 0)	780	189540	727 83360

 – subgroups

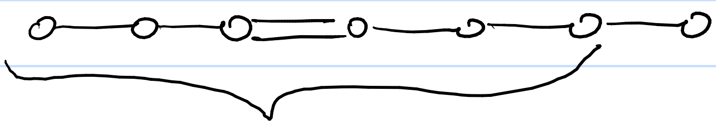
← in Y_{442}

← in Y_{333} (next page)

$\cong L_4(3).2$

Further extensions (cont.) (W.I.P.)

Twisting $Y_{333} \cong 2^3 \cdot {}^2E_6(2)$, extending Γ_1:



(computer, using ACE)

12,673,024 cosets, 4 times the index of Ω in $F_4(2)$

- Twisting Y_{443} and Y_{444} – open!
- Proving universality (without computer – open (should be the same as Y_{443})
- (A.A.Ivanov, R. Parker)
Understand Y_{332} by looking at factors of $\tilde{E}_6$ and $\tilde{E}_7$ involved there – original inspiration for this work!

Thank you!