Highly regular graphs (part 1)

Christian Pech

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(joint work with Maja Pech)

k-Homogeneity

Homogeneity

Every isomorphism between two (small) substructures of a structure can be extended to an automorphism of the structure.

The formal definition for graphs

Let $\Gamma = (V, E)$ be a graph. Γ is called k-homogeneous if for all $V_1, V_2 \subseteq V$ with $|V_1| = |V_2| \leq k$ and for each isomorphism $\psi : \Gamma(V_1) \to \Gamma(V_2)$ there exists an automorphism φ of Γ such that $\varphi\restriction_{V_1} = \psi$.

The Hierarchy of k-homogeneous graphs

Theorem (Cameron)

If a finite graph is 5-homogeneous, then it is homogeneous.

Remarks

- all homogeneous graphs are known (Gardiner (1976), Gol'fand, Klin (1978)),
- up to complement the Schläfli graph is the only 4-homogeneous, 2-connected graph that is not homogeneous (Buczak (1980), Cameron (1980)),
- all 3-homogeneous graphs are known (Cameron, Macpherson 1985),
- all 2-homogeneous graphs are known (Kantor, Liebler 1982, Liebeck, Saxl 1986),
- the latter three results rely on the Classification of Finite Simple Groups.

Graph types

Definition

A graph type $\mathbb{T}$ is a triple (Δ, ι, Θ), where

- Δ, Θ are graphs,
- $\iota \colon \Delta \hookrightarrow \Theta$ is a graph-embedding.

Order of a graph type

The order of a graph type $\mathbb{T} = (\Delta, \iota, \Theta)$ is the pair (n, m), where

- n is the order of Δ,
- m is the order of Θ.

$\mathbb{T}$-regularity

Given:

- A graph Γ,
- a graph type $\mathbb{T} = (\Delta, \iota, \Theta)$

Counting $\mathbb{T}$:

- Let $\kappa\colon \Delta \hookrightarrow \Gamma$.
- $\#(\Gamma, \mathbb{T}, \kappa)$ is the number of embeddings $\hat{\kappa}\colon \Theta \hookrightarrow \Gamma$ that make the following diagram commutative:

$$\begin{tikzcd} \Delta \arrow[r, hook, "\kappa"] \arrow[d, hook, "\iota"'] & \Gamma \\ \Theta \arrow[ur, hook, "\hat{\kappa}"'] & \end{tikzcd}$$

Remark

If ι is the identical embedding, then $\#(\Gamma, \mathbb{T}, \kappa)$ counts the number of extensions of κ to Θ.

$\mathbb{T}$-regularity (cont.)

$\mathbb{T}$-regularity

Γ is called $\mathbb{T}$-regular if $\#(\Gamma, \mathbb{T}, \iota)$ does not depend ι.
In this case this number is denoted by $\#(\Gamma, \mathbb{T})$

Example

If $\mathbb{T}$ is given by

$$x \ \circ \qquad \xhookrightarrow{=} \qquad x \ \circ\!\!-\!\!\!-\!\!\!-\!\!\circ \ y$$

then Γ is $\mathbb{T}$-regular if and only if it is regular.

Isomorphic graph types

Observation

Given

- $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$, $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$,
- isomorphisms $\varphi\colon \Delta_1 \to \Delta_2, \qquad \psi\colon \Theta_1 \to \Theta_2$,
  such that the following diagram is commutative:

$$\begin{tikzcd} \Delta_1 \arrow[r, hook, "\iota_1"] \arrow[d, "\varphi"'] & \Theta_1 \arrow[d, "\psi"] \\ \Delta_2 \arrow[r, hook, "\iota_2"] & \Theta_2. \end{tikzcd}$$

Then $\mathbb{T}_1$-regularity is the same as $\mathbb{T}_2$-regularity.

Conclusions

- For $\mathbb{T} = (\Delta, \iota, \Theta)$ we can assume w.l.o.g. that ι is the identical embedding.
- $\mathbb{T}$ is determined (up to isomorphism) by Θ and the image of ι.

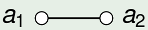
Visualization of graph types

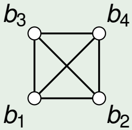
Graphical representation of $\mathbb{T} = (\Delta, \iota, \Theta)$

- Draw Θ (as an unlabelled graph)
- Color all vertices of Θ that are in the image of ι (the others leave uncolored).

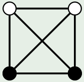
Example

Consider the graph type $\mathbb{T} = (\Delta, \iota, \Theta)$ given by:

Δ: 

Θ: 

$\iota \colon a_1 \mapsto b_1,\ a_2 \mapsto b_2$

Visualization of $\mathbb{T}$: 

(n, m)-regularity

Definition

A graph Γ is (n, m)-regular if for all $k \leq n$, $l \leq m$, and for every type $\mathbb{T}$ of order (k, l) we have that Γ is $\mathbb{T}$-regular.

- $(1, 2)$-regular is the same as regular,
- $(2, 3)$-regular is the same as strongly regular,
- $(k, k + 1)$-regular is the same as k-isoregular,
- $(2, t)$-regular is the same as fulfilling the t-vertex condition

Known examples

Hestenes, Higman (1971): Point graphs of generalized quadrangles are $(2, 4)$-regular,

A.V.Ivanov (1989): found a $(2, 5)$-regular graph on 256 vertices, whose subconstituents are $(2, 4)$-regular,

Brouwer, Ivanov, Klin (1989): generalization to an infinite series of $(3, 4)$-regular graphs whose first subconstituents are $(2, 4)$-regular,

A.V.Ivanov (1994): another infinite series of $(2, 4)$-regular graphs,

Reichard (2000): the graphs of both series are $(2, 5)$-regular,

Faradžev, A.A.Ivanov, Klin (1984) constructed a srg on 280 vertices with $\mathrm{Aut}(J_2)$ as automorphism group,

Reichard (2000): this graph is $(2, 4)$-regular,

Reichard (2003): point graphs of $\mathrm{GQ}(s, t)$ are $(2, 5)$-regular,

Reichard (2003): point graphs of $\mathrm{GQ}(q, q^2)$ are $(2, 6)$-regular,

Klin, Meszka, Reichard, Rosa (2003): the smallest $(2, 4)$-regular graph that is not 2-homogeneous has parameters $(v, k, \lambda, \mu) = (36, 14, 4, 6)$,

CP (2004): point graphs of partial quadrangles are $(2, 5)$-regular,

Reichard (2005): point graphs of $\mathrm{GQ}(q, q^2)$ are $(2, 7)$-regular,

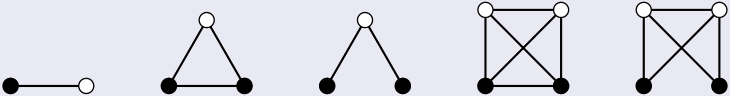
CP (2007): point graphs of $\mathrm{PQ}(q-1, q^2, q^2-q)$ are $(2, 6)$-regular,

Klin, CP (2007): found two self-complementary $(2, 4)$-regular graphs.

Some classical results

Theorem (Hestenes, Higman 1971)

A graph is $(2,4)$-regular if and only if is regular for the following graph types :

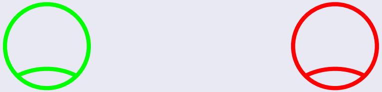


Theorem (Reichard 2000)

Let Γ be $(k, k+1)$-regular and $(2, t-1)$-regular ($t > 3$). Then, Γ is $(2, t)$-regular iff it is $\mathbb{T}$-regularity for graph types $\mathbb{T} = (\Delta, \iota, \Theta)$ of order $(2, t)$ such that all vertices of Θ that are not in the image of ι have valency $\geq k+1$.

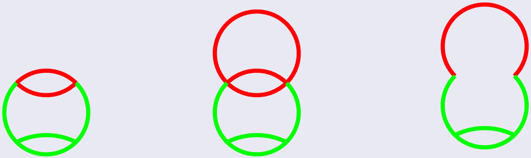
Composing types (intuitively)

Given:



$\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$

Chose a copy of Δ_2 in Θ_1 and glue the graph types together



$\mathbb{T}_1 \oplus_e \mathbb{T}_2$

Composing types (formally)

Given:

$\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$, $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$, $e \colon \Delta_2 \hookrightarrow \Theta_1$.

Consider the following diagram:

$$\begin{tikzcd}
\Theta_2 & \\
\Delta_2 \arrow[u, hook, "\iota_2"] \arrow[r, hook, "e"] & \Theta_1 \\
& \Delta_1 \arrow[u, hook, "\iota_1"']
\end{tikzcd}$$

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$\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$, $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$, $e\colon \Delta_2 \hookrightarrow \Theta_1$.

Consider the following diagram:

$$\begin{tikzcd}
\Theta_2 \arrow[r, hook, "\lambda_2"] & \Lambda \\
\Delta_2 \arrow[u, hook, "\iota_2"] \arrow[r, hook, "e"] \arrow[ur, phantom, "\urcorner"] & \Theta_1 \arrow[u, hook, "\lambda_1"'] \\
& \Delta_1 \arrow[u, hook, "\iota_1"']
\end{tikzcd}$$

Let Λ be a pushout.

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& \Delta_1 \arrow[u, hook, "\iota_1"']
\end{tikzcd}$$

Let Λ be a pushout.
Then $(\Delta_1, \lambda_1 \circ \iota_1, \Lambda)$ is a graph type. It is denoted by $\mathbb{T}_1 \oplus_e \mathbb{T}_2$.

The Type-Counting Lemma

Given:

- $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$, $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$, $e : \Delta_2 \hookrightarrow \Theta_1$,
- $\mathbb{T}_1 \oplus_e \mathbb{T}_2 = (\Delta_1, \Lambda, \lambda_1 \circ \iota_1)$.

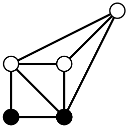
Lemma

A graph Γ is $\mathbb{T}_1 \oplus_e \mathbb{T}_2$-regular if

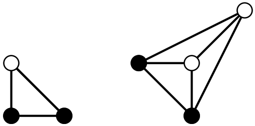
1. *Γ is $\mathbb{T}_1$-regular,*
2. *Γ is $\mathbb{T}_2$-regular, and*
3. *Γ is $\mathbb{T}$-regular for every $\mathbb{T} = (\Delta_1, \iota, \Theta)$, such that either $|V(\Theta)| < |V(\Lambda)|$ or $|V(\Theta)| = |V(\Lambda)|$ and $|E(\Lambda)| < |E(\Theta)|$.*

Example

Suppose, we want to count the graph type (Δ, ι, Θ):



This graph type decomposes as



Example (cont.)

If $\kappa\colon \Delta \hookrightarrow \Gamma$ is given, extensions of κ to Θ are constructed in two steps:

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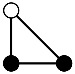
(1) : 

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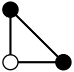
(1) : 

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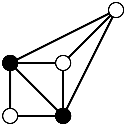
(1) : 

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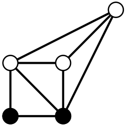
(1) : 

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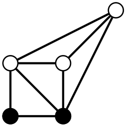
(1) : 

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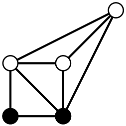
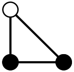
(1) : 

(2) : 

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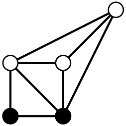
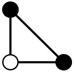
(1) : 

(2) : 

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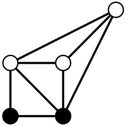
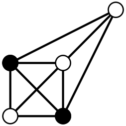
(1) : 

(2) : 

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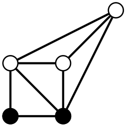
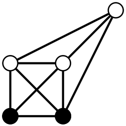
$(1):$ 

$(2):$ 

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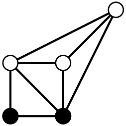

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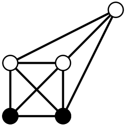

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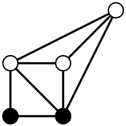
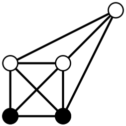
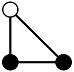
(1) : 

(2) : 

(3) : 

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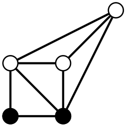

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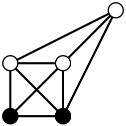

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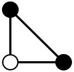

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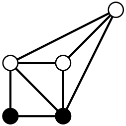
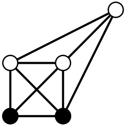
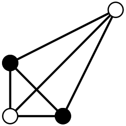
(1) : 

(2) : 

(3) : 

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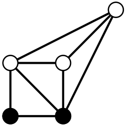

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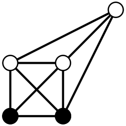

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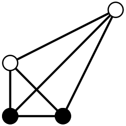

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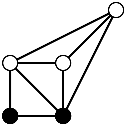

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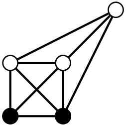

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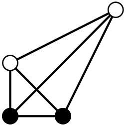

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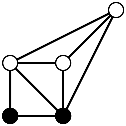

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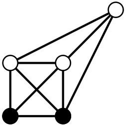

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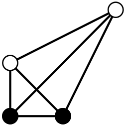

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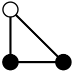

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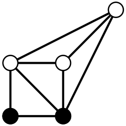
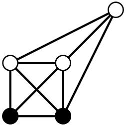
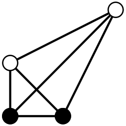
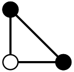
(1) : 

(2) : 

(3) : 

(4) : 

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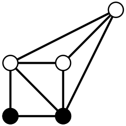
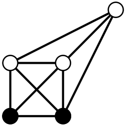
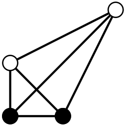
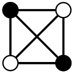
(1) : 

(2) : 

(3) : 

(4) : 

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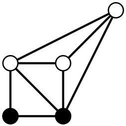
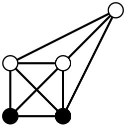
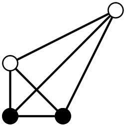
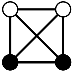
(1) : 

(2) : 

(3) : 

(4) : 

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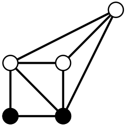
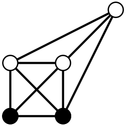
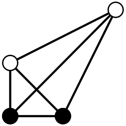
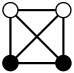
(1) : 

(2) : 

(3) : 

(4) : 

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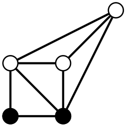

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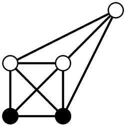

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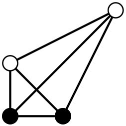

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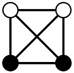

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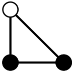

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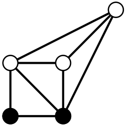
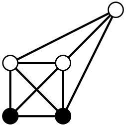
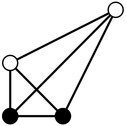
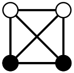
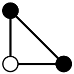
(1) : 

(2) : 

(3) : 

(4) : 

(5) : 

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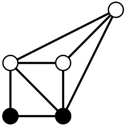
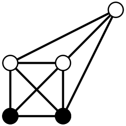
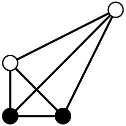
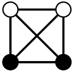
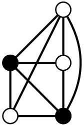
(1) : 

(2) : 

(3) : 

(4) : 

(5) : 

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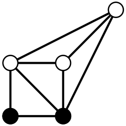

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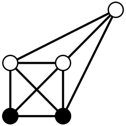

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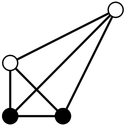

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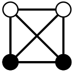

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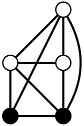

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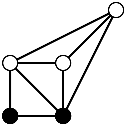
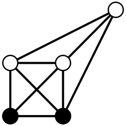
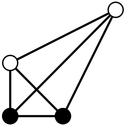
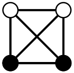
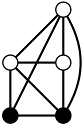
(1) : 

(2) : 

(3) : 

(4) : 

(5) : 

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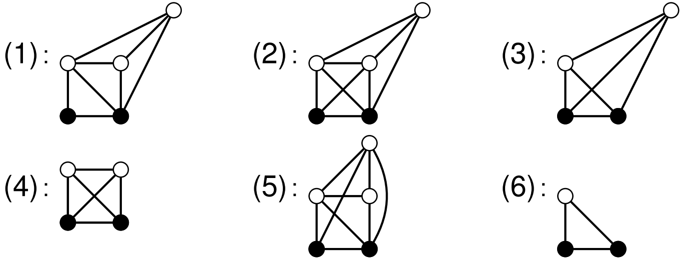

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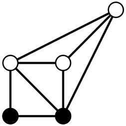

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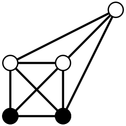

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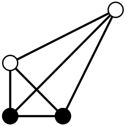

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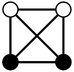

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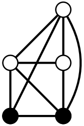

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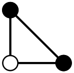

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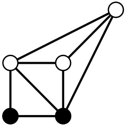
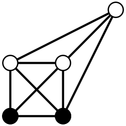
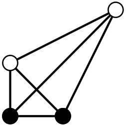
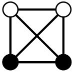
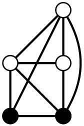
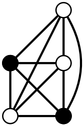
(1) : 

(2) : 

(3) : 

(4) : 

(5) : 

(6) : 

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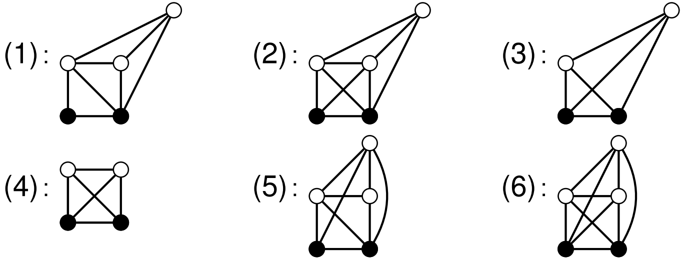

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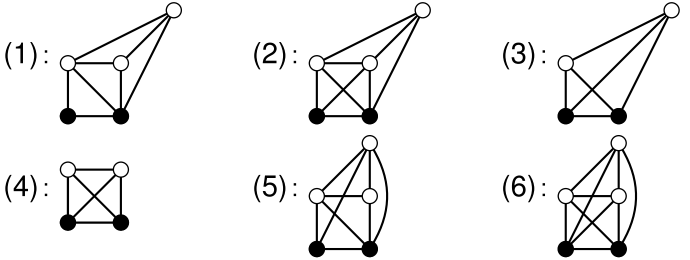
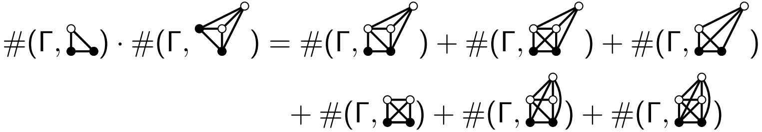


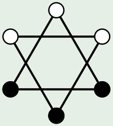
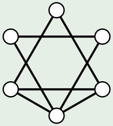


First consequence of the type-counting lemma

Definition

Given $\mathbb{T} = (\Delta, \iota, \Theta)$. Suppose $\Theta = (T, E)$. Let $S = \mathrm{Im}(\iota)$. Then $\mathrm{Cl}(\mathbb{T})$ is the graph with vertex set T and with edge set $E \cup \binom{S}{2}$.

Example

$\mathbb{T}$:  $\mathrm{Cl}(\mathbb{T})$: 

Proposition

Let Γ be (n, m)-regular (for $m > n$). Then, Γ is $(n, m+1)$-regular iff it is $\mathbb{T}$-regular for all graph-types $\mathbb{T}$ of order $(n, m+1)$ for which $\mathrm{Cl}(\mathbb{T})$ is $(n+1)$-connected.

More examples of highly regular graphs

Theorem

The point graphs of $\mathrm{GQ}(q, q^2)$ *are* $(3, 7)$*-regular.*

Smallest known $(3, 7)$-regular graph

- The smallest non-classical example is $\mathrm{GQ}(5, 25)$.
- Its point-graph has parameters $(v, k, \lambda, \mu) = (756, 130, 4, 26)$.
- its automorphism group acts intransitively on the vertices.

Final remarks

- In the second part of this talk we will identify two series of $(3, 5)$-regular graphs.
- We think that for every $t > 3$ there exists a $(3, t)$-regular graph that is not 3-homogeneous.
- We will be surprised if there exists any $(4, 6)$-regular graph that is not 4-homogeneous. The McLaughlin graph is the only known graph that is $(4, 5)$-regular but not 4-homogeneous.

Problems

Problem 1

Find more examples of highly regular graphs.

Problem 2

Is there some t_0, such that every $(4, t_0)$-regular graph is 4-homogeneous?

Problem 3

Is there some t_0, such that every $(3, t_0)$-regular graph is 3-homogeneous?

Problem 4

Is there some t_0, such that every $(2, t_0)$-regular graph is 2-homogeneous?

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