Highly regular graphs

Part two: On a family of strongly regular graphs by Brouwer, Ivanov and Klin

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It is about regularity

Graph type

A **graph type** $\mathbb{T}$ of order (m, n) is a triple (Δ, ι, Θ), where

- Δ and Θ graphs of order m and n, respectively, and
- ι is a graph embedding.

$\mathbb{T}$-regular graphs

Γ is **$\mathbb{T}$-regular** if for all $\kappa : \Delta \hookrightarrow \Theta$ the number of $\hat{\kappa} : \Theta \hookrightarrow \Gamma$ with $\kappa = \hat{\kappa} \circ \iota$ is a constant $\#(\Gamma, \mathbb{T})$, independent of κ.

Remark

W.l.o.g., in a type $\mathbb{T} = (\Delta, \iota, \Gamma)$ we may assume that ι is the identical embedding, i.e. Δ is an induced subgraph of Θ.

It is about regularity (cont.)

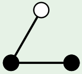
How to think about it?

(Δ, ι, Θ) may be depicted by drawing a diagram for Θ and then marking those vertices of Θ that are in $\mathrm{Im}(\iota)$.

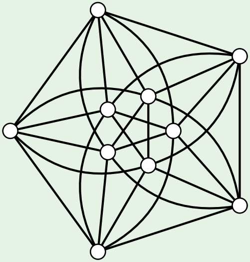
Example

$\mathbb{T}_1$:

$\mathbb{T}_2$:



Γ :



$\#(\Gamma, \mathbb{T}_1) = 6$
$\#(\Gamma, \mathbb{T}_2) = 2$

(m, n)-regular graphs

Definition

Γ is called **(m, n)-regular** if it is $\mathbb{T}$-regular for all graph types $\mathbb{T}$ of order (k, l), where $k \leq m$, $l \leq n$.

Known regularity conditions

- $(1, 2)$-regular $\equiv$ **regular**
- $(2, 3)$-regular $\equiv$ **strongly regular**
- $(2, t)$-regular $\equiv$ **t-vertex condition** (pairweise t-regular)
- $(k, k + 1)$-regular $\equiv$ **k-isoregular**

Homogeneity vs. regularity

Definition

Γ is ***k*-homogeneous** if every isomorphism between subgraphs of order $\leq k$ extends to an automorphism.

Observation

If Γ is k-homogeneous, then it is (k, l)-regular, for every $l \geq k$.

Highly regular graphs

We call Γ **highly regular** if it is

- (m, n)-regular, for some $m \geq 2$, $n \geq 4$, $m < n$, but
- not m-homogeneous.

What is known about highly regular graphs?

Almost nothing - there are very few known highly regular graphs!

Mainly:

- point graphs of partial quadrangles;
- some infinite families of graphs, uncovered by Ivanov, and Brouwer, Ivanov, Klin;
- some sporadic examples.

The Brouwer-Ivanov-Klin graphs $\Gamma^{(m)}$

Classical construction

- Consider the vector space $\mathbb{F}_2^{2m}$.
- Let $q_m : \mathbb{F}_2^{2m} \to \mathbb{F}_2$ be a non-degenerate quadric form of maximal Witt index.
- Let Q_m be the quadric defined by q_m.
- Let S_m be a maximal singular subspace of Q_m.
- Define

$$\Gamma^{(m)} := (V^{(m)}, E^{(m)}),$$

where

$$V^{(m)} = \mathbb{F}_2^{2m},$$
$$E^{(m)} = \{(\bar{v}, \bar{w}) \mid \bar{w} - \bar{v} \in Q_m \setminus S_m\}.$$

Basic facts

Definition

Given Γ, and a $v \in V(\Gamma)$. Then

$$\Gamma_1(v) := \langle \{ w \in V(\Gamma) \mid (v,w) \in E(\Gamma) \} \rangle_\Gamma \qquad \textbf{first subconstituent}$$
$$\Gamma_2(v) := \langle \{ w \in V(\Gamma) \setminus \{v\} \mid (v,w) \notin E(\Gamma) \} \rangle_\Gamma \qquad \textbf{second subconstituent}$$

First simple observations

- If Γ is vertex- transitive, then Γ has, up to isomorphism, just one first, and just one second subconstituent.
- $\Gamma^{(m)}$ is a Cayley graph w.r.t. $(\mathbb{F}_2^{2m}, +)$.
- In particular, $\Gamma^{(m)}$ is vertex-transitive.

Timeline of $\Gamma^{(m)}$

- $\Gamma^{(m)}$ is introduced. [($\Gamma^{(4)}$ by Ivanov), Brouwer, Ivanov, Klin 1989]
- $\Gamma_1^{(m)}$ is $(2,4)$-regular. [Brouwer, Ivanov, Klin 1989]
- $\Gamma_1^{(m)}$ is $(3,4)$-regular. [Brouwer, Ivanov, Klin 1989]
- For $m \geq 4$, Aut $\left(\Gamma_1^{(m)}\right)$ has rank 4. [Brouwer, Ivanov, Klin 1989]
  (In particular, $\Gamma_1^{(m)}$ is NOT 2-homogeneous.)
- $\Gamma^{(m)}$ is symmetric. [Ivanov 1990]
  (Aut $\left(\Gamma^{(m)}\right)$ acts transitively on arcs.)
- $\Gamma^{(m)}$ is $(2,5)$-regular. [Reichard 2000]

Still unknown

Question

Are $\Gamma^{(m)}$ or $\Gamma_2^{(m)}$ 2-homogeneous?

Open problem

Is $\Gamma_2^{(m)}$ $(2, 4)$-regular?

Towards the answer

Some thoughts, collected on a margin

- For a detailed analysis of the $\Gamma^{(m)}$, and for its implementation in GAP a more direct construction is needed.
- Every non-degenerate quadric form on $\mathbb{F}_2^{2m}$ has Witt index $\leq m$.
- There is, up to equivalence, just one non-degenerative form of Witt index m:

$$q_m(x_1, \ldots, x_m, y_1, \ldots, y_m) = \sum_{i=1}^{m} x_i y_i.$$

Let's have another look on $\Gamma^{(m)}$

Our way to construct it

- Every $\bar{x} \in \mathbb{F}_2^{2m}$ can be considered as $\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2, \end{pmatrix}$ where $\bar{x}_1, \bar{x}_2 \in \mathbb{F}_2^m$.
- With this convention we have

$$
\begin{aligned}
q_m(\bar{x}) =& \bar{x}_1^T \bar{x}_2, \\
Q_m =& \{\bar{x} \mid \bar{x}_1^T \bar{x}_2 = 0\}, \\
S_m =& \{\bar{x} \mid \bar{x}_2 = \bar{0}\}, \\
E(\Gamma^{(m)}) =& \{(\bar{x}, \bar{y}) \mid (\bar{x}_1 + \bar{y}_1)^T (\bar{x}_2 + \bar{y}_2) = 0, \bar{x}_2 \neq \bar{y}_2\}.
\end{aligned}
$$

What brings us this point of view?

Lemma

Let $M \in \mathrm{GL}(2m, 2)$. Then M preserves Q_m and S_m setwise iff there exist some $A \in \mathrm{GL}(m, 2)$, and some symmetric $m \times m$-matrix S with 0-diagonal, such that

$$M = \begin{pmatrix} A & AS \\ O & (A^T)^{-1} \end{pmatrix}.$$

What brings us this point of view? (cont.)

Proposition

Define

$$\begin{aligned}
\varrho_1^{(m)} &:= \{(\bar{v}, \bar{w}) \mid \bar{v} = \bar{w}\},\\
\varrho_2^{(m)} &:= \{(\bar{v}, \bar{w}) \mid \bar{v} + \bar{w} \in S_m \setminus \{\bar{0}\}\},\\
\varrho_3^{(m)} &:= \{(\bar{v}, \bar{w}) \mid \bar{v} + \bar{w} \in Q_m \setminus S_m\},\\
\varrho_4^{(m)} &:= \{(\bar{v}, \bar{w}) \mid \bar{v} + \bar{w} \in \mathbb{F}_2^{2m} \setminus Q_m\}.
\end{aligned}$$

Then $\mathcal{C}^{(m)} = (\mathbb{F}_2^{2m}, \varrho_1^{(m)}, \varrho_2^{(m)}, \varrho_3^{(m)}, \varrho_4^{(m)})$ is a Schurian symmetric association scheme.

Moreover, considered as a relational structure, it is 3-homogeneous, i.e. every isomorphism between substructures with ≤ 3 elements extends to an automorphism of $\mathcal{C}^{(m)}$.

What brings us this point of view? (cont.)

Corollary

Either $\Gamma^{(m)}$ is 2-homogeneous or the orbitals of $\mathrm{Aut}(\Gamma^{(m)})$ are exactly

$$\varrho_1^{(m)}, \varrho_2^{(m)}, \varrho_3^{(m)}, \varrho_4^{(m)}.$$

Questions are there to be answered...

Proposition

For $m \geq 4$, $\Gamma^{(m)}$ is not 2-homogeneous and $\mathrm{Aut}(\Gamma_2^{(m)})$ is intransitive.

Sketch of the proof

- Consider non-arcs $(\bar{0}, \bar{a})$, $(\bar{0}, \bar{b})$, where $\bar{a} \in S_m \setminus \{\bar{0}\}$, $\bar{b} \in \mathbb{F}_2^{2m} \setminus Q_m$.
- Define

$$\Upsilon_{\bar{a}} := \left\langle \left\{ \bar{x} \mid (\bar{x}, \bar{0}) \in E(\Gamma^{(m)}), (\bar{x}, \bar{a}) \notin E(\Gamma^{(m)}) \right\} \right\rangle_{\Gamma^{(m)}}$$

$$\Upsilon_{\bar{b}} := \left\langle \left\{ \bar{x} \mid (\bar{x}, \bar{0}) \in E(\Gamma^{(m)}), (\bar{x}, \bar{b}) \notin E(\Gamma^{(m)}) \right\} \right\rangle_{\Gamma^{(m)}}$$

- Then $\Upsilon_{\bar{a}} \cong \Gamma^{(m-1)}$, $\Upsilon_{\bar{b}}$ is $(2,3)$-regular, but not $(3,4)$-regular.
- Hence $\Upsilon_{\bar{a}} \not\cong \Upsilon_{\bar{b}}$
- Now $\Upsilon_{\bar{a}} = \left(\Gamma_2^{(m)}(\bar{a})\right)_1 (\bar{0})$, and $\Upsilon_{\bar{b}} = \left(\Gamma_2^{(m)}(\bar{b})\right)_1 (\bar{0})$.
- Since $\Gamma_2^{(m)}(\bar{a}) \not\cong \Gamma_2^{(m)}(\bar{b})$, it follows that $\mathrm{Aut}(\Gamma_2^{(m)})$ is intransitive. $\square$

. . . and what is open should be closed

Theorem

For $m \geq 4$, $\Gamma^{(m)}$ is $(3,5)$-regular.

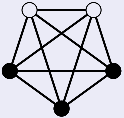
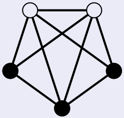
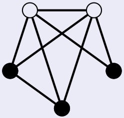
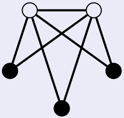
Corollary

For $m \geq 4$, $\Gamma_2^{(m)}$ is $(2,4)$-regular, but not 2-homogeneous.

. . . and what is open should be closed (cont.)

Sketch of the proof

- Since $\Gamma^{(m)}$ is $(3,4)$-regular, it suffices to check regularity for graph types $\mathbb{T}$ of order $(3,5)$ for which $\mathrm{Cl}(\mathbb{T})$ is 4-connected.
- The only 4-connected graph of order 5 is K_5.
- Thus, regularity for the following types has to be proved:

$\mathbb{T}_1$:  $\mathbb{T}_2$:  $\mathbb{T}_3$:  $\mathbb{T}_4$: 

- $\mathbb{T}_1$-regularity follows from 3-homogeneity of $\mathcal{C}^{(m)}$.
- $\mathbb{T}_2$-regularity follows from $(2,4)$-regularity of $\Gamma_1^{(m)}$.

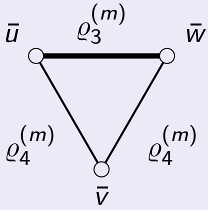
. . . and what is open should be closed (cont.)

Sketch of the proof (cont.)

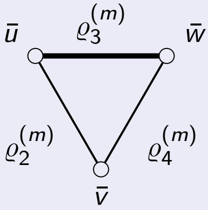
$\mathbb{T}_3$-regularity and $\mathbb{T}_4$-regularity is proved by case distinction:
About $\mathbb{T}_3 = (\Delta, \iota, \Theta)$. Let $\kappa : \Delta \hookrightarrow \Gamma$.

- Two kids of embedding κ are to be distinguished:

(1) :



(2) :

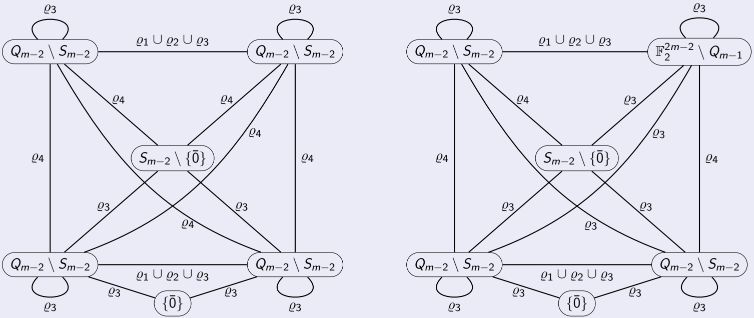


- In each case we need to count arcs in the subgraph of $\Gamma^{(m)}$ induced by the joint neighbors of $\bar{u}$, $\bar{v}$, $\bar{w}$.

. . . and what is open should be closed (cont.)

Sketch of the proof (cont.)

- This can be done by studying their "reflections" in $\mathcal{C}^{(m-2)}$:



- Now, arcs can be counted using the structure constants of $\mathcal{C}^{(m-2)}$.

$\square$