Schurian vs non-schurian coherent configurations

Ilia Ponomarenko

St.Petersburg Department of V.A.Steklov Institute of Mathematics
of the Russian Academy of Sciences

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The Weisfeiler-Leman algorithm (1968)

Let X be a graph with vertex set Ω, $n = |\Omega|$.

The WL-algorithm

1. Set initial coloring: $c(\alpha, \beta) = 0, 1$, or 2 depending on (α, β) is loop, arc, or neither loop nor arc, for all $(\alpha, \beta) \in \Omega \times \Omega$.
2. Find a multiset
$$s(\alpha, \beta) = \{\{s_\gamma(\alpha, \beta) : \gamma \in \Omega\}\}, \quad s_\gamma(\alpha, \beta) = (c(\alpha, \gamma), c(\gamma, \beta)).$$
3. Define a coloring c': $c'(\alpha, \beta) < c'(\gamma, \delta) \Leftrightarrow s(\alpha, \beta) \prec s(\gamma, \delta)$.
4. Go to step 2 if $|c| \neq |c'|$; otherwise, output c.

Remarks

- The vertex coloring is given by $c(\alpha, \alpha)$, $\alpha \in \Omega$.
- The running time is $O(n^3 \log n)$ (Immerman-Lander, 1990).
- The WL-algorithm fails for strongly regular graphs with intransitive automorphism group.

Coherent configurations

Observation

The partition S of $\Omega \times \Omega$ into the color classes of the coloring obtained by the WL-algorithm forms a coherent configuration.

A pair (Ω, S) is a coherent configuration (cc) if

- $1_\Omega = \{(\alpha,\alpha) : \ \alpha \in \Omega\}$ is a union of relations from S,
- S contains $s^* = \{(\alpha,\beta) : \ (\beta,\alpha) \in s\}$ for all $s \in S$,
- for all $r, s, t \in S$, the intersection number

$$c_{rs}^t = |\{\gamma \in \Omega : \ (\alpha,\gamma) \in r, \ (\gamma,\beta) \in s\}|$$

  does not depend on the choice of $(\alpha,\beta) \in t$.

Example:

A cc is schurian if $S = \mathrm{Orb}(G, \Omega \times \Omega)$ for a group $G \leq \mathrm{Sym}(\Omega)$.

Small coherent configurations are schurian

Leman, 1970

Any cc with $n \leq 7$ is schurian.

the Hanaki–Miyamoto list, $\cong$1990s

- any homogeneous cc with $n \leq 14$ is schurian,
- there is a unique non-schurian homogeneous cc with $n = 15$.

Klin–Ziv-Av, 2016

- any cc with $n \leq 13$ is schurian,
- there is a unique non-schurian cc with $n = 14$.

Evdokimov–Kovács–P, 2013

- any S-ring over C_n, $n \leq 71$, is schurian,
- there are non-schurian S-rings over C_{72}.

Several conjectures on schurity

Wielandt, 1960s

- "Schur had conjectured for a long time that every S-ring is determined by a suitable permutation group",
- a family of non-schurian S-rings over $C_p \times C_p$ (W, 1955).

Weisfeiler–Leman, 1968-1969

- conjecture: the fibers of a cc $\mathcal{X}$ are the orbits of $\mathrm{Aut}(\mathcal{X})$;
- a non-schurian cc, $n = 26$ (Adel'son-Vel'ski–Faradžev–L–W).

Friedland, 1989

- all cc with simple spectrum are schurian,
- a counterexample, $n = 16$ (Evdokimov, 1990s).

Prolific non-schurian families

Geometric and combinatorial examples

- non-Desarguesian projective planes (D. Higman, 1970s),
- Latin squares of finite groups (Enomoto, 1971).

Affine schemes

Let $\mathcal{X}$ be the cc of $G \leq \mathrm{AGL}_2(p) \leq \mathrm{Sym}(p^2)$, formed by p^2 translations and the center of $\mathrm{GL}_2(p)$ ($|G| = p^2(p-1)$). Then

- $|S| = p + 2$ and any subset of $\{1, \ldots, p+1\}$ produces a fusion of $\mathcal{X}$ (Gol'fand–Klin, 1985);
- there are exactly 2^{p+1} of such fusions, and only $\mathrm{poly}(p)$ of them are schurian;
- each fusion is isomorphic to at most $|\,\mathrm{GL}_2(p)|$ other fusions (Godsil, 1983),
- thus there are at least $2^{O(p)}$ nonisomorphic non-schurian cc with $n = p^2$.

The schurity problem: statement

The Graph Isomorphism Problem and schurian cc

Let $\mathcal{K}$ be a class of graphs closed with respect to disjoint union. If the cc of any graph from $\mathcal{K}$ is schurian, then the isomorphism of two n-vertex graphs from $\mathcal{K}$ can be tested in time poly(n).

Problem A

Determine schurian cc in a given class of cc.

Examples

- the classification of spherical buildings of finite order solves the Problem A for the Coxeter schemes (Tits, 1974),
- the classification of the rank 3 groups solves the Problem A for the symmetric schemes of rank 3 (BKLPS..., 1980s),
- the classification of distance-transitive graphs (not yet done).

The schurity problem: Cayley schemes, 1

Definition

A cc $\mathcal{X}$ is called a Cayley scheme over a group G if

$$\Omega = G \quad \text{and} \quad \mathrm{Aut}(\mathcal{X}) \geq G_{right}$$

If G is cyclic, $\mathcal{X}$ is said to be circulant.

Observation

The schurity problem for circulant schemes reduces to quasidence case, i.e., for those $\mathcal{X}$ such that given an $\mathcal{X}$-section S of G:

$$\mathcal{X}_S = (S, \mathrm{Orb}(K)) \quad \text{with } S \leq K \leq S\,\mathrm{Aut}(S).$$

Moreover, the schemes $\mathcal{X}_S$ must be pairwise compatible.

Theorem (Evdokimov–P, 2015)

A quasidense circulant scheme is schurian if and only if a suitable modular linear system (with unknowns x_S) has a solution.

The schurity problem: Cayley schemes, 2

Definition (Pöschel, 1974)

G is a Schur group if every Cayley scheme over G is schurian.

Examples

- E_n is Schur $\Leftrightarrow$ $n \in \{4, 8, 9, 16, 27, 32\}$.
- C_n is Schur $\Leftrightarrow$ $n = p^k$, pq^k, $2pq^k$, pqr, $2pqr$ (EKP, 2012),
- a Schur group G is metabelian and $|\pi(G)| \leq 7$ (PV, 2014).

Theorem (Evdokimov-Kovács-P, 2012)

Every abelian Schur group other than E_n or C_n is as follows:

- $C_p \times C_{p^k}$ for $p = 2$ or $p = 3$,
- $C_{2p} \times C_{2^k}$, $E_4 \times C_{p^k}$, $E_4 \times C_{pq}$, $E_{16} \times C_p$ for $p \neq 2$,
- $C_6 \times C_{3^k}$, $E_9 \times C_{2q}$, $E_9 \times C_p$ for $p \neq 2$.

The schurity problem: amalgams

Definition

$\mathcal{X}$ is called a simple spectrum cc if each homogeneous component of $\mathcal{X}$ is an abelian group. In the schurian case,

$$\mathrm{Aut}(\mathcal{X}) \leq G_1 \times \cdots \times G_m,$$

where G_i is a regular abelian group and m is the number of fibers.

Observation

The axioms of cc imply that there exist a family $\{G_{ij}\}_{i,j=1}^{m}$ of subgroups $G_{ij} \leq G_i$ such that

$$G_i/G_{ij} \cong G_j/G_{ji}, \quad 1 \leq i, j \leq m.$$

Theorem (Hirasaka-Kim-P, 2018)

$\mathcal{X}$ is schurian $\Leftrightarrow$ there exists the generalized free product of the groups $\widehat{G}_i$ with amalgamated subgroups $\widehat{G}_{ij}$.

The schurity problem: the Desargues theorem

A non-commutative geometry

For a homogeneous cc $\mathcal{X}$, we define a non-commutative geometry:

- the points are elements of S,
- the lines are the sets x^*y, $x, y \in S$,
- the incidence relation is given by inclusion.

Theorem (Hirasaka–Kim–P, in preparation)

A two-valenced scheme $\mathcal{X}$ with "many" intersection numbers ≤ 1, is schurian whenever the geometry of $\mathcal{X}$ is Desarguesian.

Corollary

There are combinatorial characterization of

- the Frobenius groups (Muzychuk–P, 2012),
- central extensions of E_{p^m}, $m \geq 3$ (Hirasaka–Kim–P),
- transitive groups with point stabilizer of order 2 (M–P, 2012).

The schurity problem: algorithms

Problem B

Given a cc $\mathcal{X}$, test whether $\mathcal{X}$ is schurian or not.

The Problem B can be solved

- in quasipolynomial time in n (Babai, 2015),
- in time $n^{O(k/logk)}$ if $\mathcal{X}$ contains a connected relation of maximal valency k (Babai–Kantor–Luks, 1983),
- in polynomial time if $\mathcal{X}$ is antisymmetric (P, 2012).

Remark

We believe that Problem B can be solved in polynomial time in the class of all cc. Two interesting special cases are: affine schemes and strongly regular graphs.

The m-dim WL-algorithm

The m-dim WL-algorithm, $m \geq 3$ goes back to "deep stabilization" in the sense of Weisfeiler-Leman and was renewed by Babai.

The m-dim WL-algorithm

1. Set initial coloring: $c(u)$ is equal to "the isomorphism type" of the graph induced on $\{u_1, \ldots, u_m\}$, $u \in \Omega^m$
2. Find a multiset $s(u) = \{\{s_\gamma(u) : \gamma \in \Omega\}\}$, where $s_\gamma(u) = (c(u_{1,\gamma}), \ldots, c(u_{m,\gamma}))$ with $u_{i,\gamma} = (u_1, \ldots, u_{i-1}, \gamma, u_{i+1}, \ldots, u_m)$.
3. Define a coloring c': $c'(u) < c'(v) \Leftrightarrow s(u) \prec s(v)$.
4. Go to step 2 if $|c| \neq |c'|$; otherwise, output c.

Remarks

- The vertex coloring is given by $c(\alpha, \ldots, \alpha)$, $\alpha \in \Omega$.
- For $m = 2$, this is exactly the WL-algorithm.
- The complexity: $O(m^2 n^{m+1} \log n)$ (Immerman–Lander, 1990).

Multidimensional closure

Two ways to define the m-dim closure of $\mathcal{X}$ for $m \geq 2$, are

$$\mathrm{WL}_m(\mathcal{X}) = \mathrm{pr}_{\Omega^2}\, \mathrm{WL}_m(\mathcal{X}^m)$$

and

$$\overline{\mathcal{X}}^{(m)} = \mathrm{pr}_{\Omega^2}\, \mathrm{WL}_2(\mathcal{X}^m, \mathrm{Diag}(\Omega^m)),$$

Theorem (Evdokimov–P, 1999-2001)

- $\mathrm{WL}_m(\mathcal{X}) \leq \overline{\mathcal{X}}^{(m)} \leq \mathrm{WL}_{3m}(\mathcal{X})$.
- $\mathcal{X} = \overline{\mathcal{X}}^{(1)} \leq \cdots \leq \overline{\mathcal{X}}^{(t)} = \cdots = \mathcal{X}^{(\infty)}$,
- $\mathcal{X}^{(\infty)}$ is schurian and $\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}(\mathcal{X}^{(\infty)})$,
- $t \leq \lfloor n/3 \rfloor$.

Theorem (Cai– Fürer–Immerman, 1992; Evdokimov–P, 1999)

There is a constant $c > 0$ and a family $\{\mathcal{X}_n\}$ of cc such that

$$|\Omega(\mathcal{X}_n)| = n \quad \text{and} \quad t(\mathcal{X}_n) \geq \lfloor cn \rfloor.$$

Multidimensional closure: examples

Projective planes

Let $\mathcal{X}$ be the scheme of a projective plane $\mathcal{P}$. Then

- $t(\mathcal{X}) = 1 \Leftrightarrow \mathcal{P}$ is a Galois plane,
- $t(\mathcal{X}) \geq 3$ if $\mathcal{P}$ is not a Galois plane (Evdokimov–P, 2010).

Planar graphs (Kiefer–P–Schweitzer, 2017)

Let $\mathcal{X}$ be the cc of a planar graph. Then $t(\mathcal{X}) \leq 2$.

Steiner t-designs (Babai–Wilmes, 2013)

Let $\mathcal{X}$ be the cc of a nontrivial Steiner design on n points. Then $t(\mathcal{X}) \leq c \log n$ for a constant $c > 0$.