Issai Schur, Helmut Wielandt and schurian Schur-rings

Reinhard Pöschel

Institut für Algebra
Technische Universität Dresden
Germany

Conference in Algebraic Graph Theory
Symmetry vs Regularity
The first 50 years since Weisfeiler-Leman stabilization
July 1 - July 7, 2018
Pilsen

Outline

Some history: Burnside - Schur - Wielandt

Some more history and the characterization of (non)-Schurian S-rings

William Burnside (2.7.1852 - 21.8.1927)

Theorem

A transitive permutation group of prime degree is doubly transitive or solvable.

On the properties of groups of odd order. Proc. London Math. Soc. XXXIII (1900)

Theorem

A permutation group of prime power degree $n = p^m$ containing a cycle of order n is either doubly transitive or imprimitive.

Theory of Groups of Finite Order. (1911)
Proof uses character theory.

W. Burnside conjectured that an analogous result holds for every permutation group of degree n that contains a regular Abelian subgroup of order n. **FALSE!** (counterexample exists) but **TRUE** if the regular subgroup is cyclic. (*I. Schur*)

William Burnside (2.7.1852 - 21.8.1927)

Theorem

A transitive permutation group of prime degree is doubly transitive or solvable.

On the properties of groups of odd order. Proc. London Math. Soc. XXXIII (1900)

Theorem

A permutation group of prime power degree $n = p^m$ containing a cycle of order n is either doubly transitive or imprimitive.

Theory of Groups of Finite Order. (1911)
Proof uses character theory.

W. Burnside conjectured that an analogous result holds for every permutation group of degree n that contains a regular Abelian subgroup of order n. **FALSE!** (counterexample exists) but **TRUE** if the regular subgroup is cyclic. (*I. Schur*)

William Burnside (2.7.1852 - 21.8.1927)

Theorem

A transitive permutation group of prime degree is doubly transitive or solvable.

On the properties of groups of odd order. Proc. London Math. Soc. XXXIII (1900)

Theorem

A permutation group of prime power degree $n = p^m$ containing a cycle of order n is either doubly transitive or imprimitive.

Theory of Groups of Finite Order. (1911)

Proof uses character theory.

W. Burnside conjectured that an analogous result holds for every permutation group of degree n that contains a regular Abelian subgroup of order n. **FALSE!** (counterexample exists) but **TRUE** if the regular subgroup is cyclic. (*I. Schur*)

William Burnside (2.7.1852 - 21.8.1927)

Theorem

A transitive permutation group of prime degree is doubly transitive or solvable.

On the properties of groups of odd order. Proc. London Math. Soc. XXXIII (1900)

Theorem

A permutation group of prime power degree $n = p^m$ containing a cycle of order n is either doubly transitive or imprimitive.

Theory of Groups of Finite Order. (1911)
Proof uses character theory.

W. Burnside conjectured that an analogous result holds for every permutation group of degree n that contains a regular Abelian subgroup of order n. **FALSE!** (counterexample exists) but **TRUE** if the regular subgroup is cyclic. (*I. Schur*)

William Burnside (2.7.1852 - 21.8.1927)

Theorem

A transitive permutation group of prime degree is doubly transitive or solvable.

On the properties of groups of odd order. Proc. London Math. Soc. XXXIII (1900)

Theorem

A permutation group of prime power degree $n = p^m$ containing a cycle of order n is either doubly transitive or imprimitive.

Theory of Groups of Finite Order. (1911)
Proof uses character theory.

W. Burnside conjectured that an analogous result holds for every permutation group of degree n that contains a regular Abelian subgroup of order n. **FALSE!** (counterexample exists) but **TRUE** if the regular subgroup is cyclic. (*I. Schur*)

William Burnside (2.7.1852 - 21.8.1927)

Theorem

A transitive permutation group of prime degree is doubly transitive or solvable.

On the properties of groups of odd order. Proc. London Math. Soc. XXXIII (1900)

Theorem

A permutation group of prime power degree $n = p^m$ containing a cycle of order n is either doubly transitive or imprimitive.

Theory of Groups of Finite Order. (1911)
Proof uses character theory.

W. Burnside conjectured that an analogous result holds for every permutation group of degree n that contains a regular Abelian subgroup of order n. **FALSE!** (counterexample exists) but **TRUE** if the regular subgroup is cyclic. (*I. Schur*)

William Burnside (2.7.1852 - 21.8.1927)

Theorem

A transitive permutation group of prime degree is doubly transitive or solvable.

On the properties of groups of odd order. Proc. London Math. Soc. XXXIII (1900)

Theorem

A permutation group of prime power degree $n = p^m$ containing a cycle of order n is either doubly transitive or imprimitive.

Theory of Groups of Finite Order. (1911)
Proof uses character theory.

W. Burnside conjectured that an analogous result holds for every permutation group of degree n that contains a regular Abelian subgroup of order n. **FALSE!** (counterexample exists) but **TRUE** if the regular subgroup is cyclic. (*I. Schur*)

William Burnside (2.7.1852 - 21.8.1927)

Theorem

A transitive permutation group of prime degree is doubly transitive or solvable.

On the properties of groups of odd order. Proc. London Math. Soc. XXXIII (1900)

Theorem

A permutation group of prime power degree $n = p^m$ containing a cycle of order n is either doubly transitive or imprimitive.

Theory of Groups of Finite Order. (1911)
Proof uses character theory.

W. Burnside conjectured that an analogous result holds for every permutation group of degree n that contains a regular Abelian subgroup of order n. **FALSE!** (counterexample exists) but **TRUE** if the regular subgroup is cyclic. (*I. Schur*)

Issai Schur (10.1.1875 - 10.1.1941)

on the tombstone:
Yeshiyahu Schur
Professor of Mathematics

4 Shevet 5635 (10.1.1875)
12 Tevet 5701 (11.1.1941)

Issai Schur (10.1.1875 - 10.1.1941)

on the tombstone:
Yeshiyahu Schur
Professor of Mathematics

4 Shevet 5635 (10.1.1875)
12 Tevet 5701 (11.1.1941 [starting 10.1., 6 p.m.])

In memoriam Issai Schur

Mikhail Klin – Andy Woldar

(Tel Aviv 1996)

Theorem

The following theorem of *I. Schur* generalizes the result of *W. Burnside* (case $n = p^m$) and partially answers his conjecture:

Theorem

Let $\mathfrak{G}$ be a permutation group of degree n where n is not a prime. If $\mathfrak{G}$ contains a cycle P of order n then $\mathfrak{G}$ is either doubly transitive or imprimitive.

I. Schur, *Zur Theorie der einfach transitiven Permutationsgruppen.* (1933)
"The proof of this theorem, which sounds so simple, turns out to be rather difficult. B u r n s i d e proves the theorem for the case of prime powers using character theory which requires working with roots of unity. I prove this theorem avoiding irrationalities of any kind. Suggested by B u r n s i d e's conjecture in the first place the general case of any regular permutation group $\mathfrak{H}$ embedded in a group $\mathfrak{G}$ of the same degree will be studied. This implies a decomposition of elements of $\mathfrak{H}$ into certain sub-complexes which I denote by primary complexes of $\mathfrak{H}$. The fact that these complexes have the "ring property" plays a major role (§2)." (p. 598)

Such complexes with "ring property" were later called *Schur-rings* and, for short, *S-rings* by *H. Wielandt* (1949, 1969)

Theorem

The following theorem of *I. Schur* generalizes the result of *W. Burnside* (case $n = p^m$) and partially answers his conjecture:

Theorem

Let $\mathfrak{G}$ be a permutation group of degree n where n is not a prime. If $\mathfrak{G}$ contains a cycle P of order n then $\mathfrak{G}$ is either doubly transitive or imprimitive.

I. Schur, *Zur Theorie der einfach transitiven Permutationsgruppen.* (1933)

"The proof of this theorem, which sounds so simple, turns out to be rather difficult. B u r n s i d e proves the theorem for the case of prime powers using character theory which requires working with roots of unity. I prove this theorem avoiding irrationalities of any kind. Suggested by B u r n s i d e's conjecture in the first place the general case of any regular permutation group $\mathfrak{H}$ embedded in a group $\mathfrak{G}$ of the same degree will be studied. This implies a decomposition of elements of $\mathfrak{H}$ into certain sub-complexes which I denote by primary complexes of $\mathfrak{H}$. The fact that these complexes have the "ring property" plays a major role (§2)." (p. 598)

Such complexes with "ring property" were later called *Schur-rings* and, for short, *S-rings* by *H. Wielandt* (1949, 1969)

Theorem

The following theorem of *I. Schur* generalizes the result of *W. Burnside* (case $n = p^m$) and partially answers his conjecture:

Theorem

Let $\mathfrak{G}$ be a permutation group of degree n where n is not a prime. If $\mathfrak{G}$ contains a cycle P of order n then $\mathfrak{G}$ is either doubly transitive or imprimitive.

I. Schur, *Zur Theorie der einfach transitiven Permutationsgruppen.* (1933)
"The proof of this theorem, which sounds so simple, turns out to be rather difficult. B u r n s i d e proves the theorem for the case of prime powers using character theory which requires working with roots of unity. I prove this theorem avoiding irrationalities of any kind. Suggested by B u r n s i d e's conjecture in the first place the general case of any regular permutation group $\mathfrak{H}$ embedded in a group $\mathfrak{G}$ of the same degree will be studied. This implies a decomposition of elements of $\mathfrak{H}$ into certain sub-complexes which I denote by primary complexes of $\mathfrak{H}$. The fact that these complexes have the "ring property" plays a major role (§2)." (p. 598)

Such complexes with "ring property" were later called *Schur-rings* and, for short, *S-rings* by *H. Wielandt* (1949, 1969)

Theorem

The following theorem of *I. Schur* generalizes the result of *W. Burnside* (case $n = p^m$) and partially answers his conjecture:

Theorem

Let $\mathfrak{G}$ be a permutation group of degree n where n is not a prime. If $\mathfrak{G}$ contains a cycle P of order n then $\mathfrak{G}$ is either doubly transitive or imprimitive.

I. Schur, *Zur Theorie der einfach transitiven Permutationsgruppen.* (1933)
"The proof of this theorem, which sounds so simple, turns out to be rather difficult. B u r n s i d e proves the theorem for the case of prime powers using character theory which requires working with roots of unity. I prove this theorem avoiding irrationalities of any kind. Suggested by B u r n s i d e's conjecture in the first place the general case of any regular permutation group $\mathfrak{H}$ embedded in a group $\mathfrak{G}$ of the same degree will be studied. This implies a decomposition of elements of $\mathfrak{H}$ into certain sub-complexes which I denote by primary complexes of $\mathfrak{H}$. The fact that these complexes have the "ring property" plays a major role (§2)." (p. 598)

Such complexes with "ring property" were later called *Schur-rings* and, for short, *S-rings* by *H. Wielandt* (1949, 1969)

Transitivity modules

Let a permutation group (G, M) (i.e., $G \leq \mathrm{Sym}(M)$) contain a regular subgroup H. Then one can assume $M = H$, i.e., $G \leq \mathrm{Sym}(H)$.

Let $\{T_0, \ldots, T_{r-1}\} := 1\text{-}\mathrm{Orb}(G_e)$ be the 1-orbits of the *stabilizer* G_e, ($e \in H$ unit element).

Let $\mathbb{Z}(H)$ denote the *group ring* $\langle \mathbb{Z}(H); +, * \rangle$ consisting of formal sums $\sum_{h \in H} \alpha_h h$, $\alpha_h \in \mathbb{Z}$.

Definition The submodule of $\mathbb{Z}(H)$ generated by the 1-orbits of the stabilizer G_e ($\underline{T} := \sum_{t \in T} t \in \mathbb{Z}(H)$, $T \subseteq H$)

$$\mathcal{S}(G, H) := \langle \underline{T_0}, \ldots, \underline{T_{r-1}} \rangle_{\mathbb{Z}}$$

is called the *transitivity module of* (G, H).

Theorem (*I. Schur* 1933): *$\mathcal{S}(G, H)$ is an S-ring.*

Contrary to the intuition of *I. Schur* not every S-ring is the transitivity module of a group (G, H)
– counterexamples by *H. Wielandt* (1954), e.g., for $H = \mathbb{Z}_5 \times \mathbb{Z}_5$.

Transitivity modules

Let a permutation group (G, M) (i.e., $G \leq \mathrm{Sym}(M)$) contain a regular subgroup H. Then one can assume $M = H$, i.e., $G \leq \mathrm{Sym}(H)$.
Let $\{T_0, \ldots, T_{r-1}\} := 1\text{-}\mathrm{Orb}(G_e)$ be the 1-orbits of the *stabilizer* G_e, ($e \in H$ unit element).

Let $\mathbb{Z}(H)$ denote the *group ring* $\langle \mathbb{Z}(H); +, * \rangle$ consisting of formal sums $\sum_{h \in H} \alpha_h h$, $\alpha_h \in \mathbb{Z}$.

Definition The submodule of $\mathbb{Z}(H)$ generated by the 1-orbits of the stabilizer G_e ($\underline{T} := \sum_{t \in T} t \in \mathbb{Z}(H)$, $T \subseteq H$)

$$\mathcal{S}(G, H) := \langle \underline{T_0}, \ldots, \underline{T_{r-1}} \rangle_{\mathbb{Z}}$$

is called the *transitivity module of* (G, H).

Theorem (*I. Schur* 1933): *$\mathcal{S}(G, H)$ is an S-ring.*

Contrary to the intuition of *I. Schur* not every S-ring is the transitivity module of a group (G, H)
– counterexamples by *H. Wielandt* (1954), e.g., for $H = \mathbb{Z}_5 \times \mathbb{Z}_5$.

Transitivity modules

Let a permutation group (G, M) (i.e., $G \leq \mathrm{Sym}(M)$) contain a regular subgroup H. Then one can assume $M = H$, i.e., $G \leq \mathrm{Sym}(H)$.
Let $\{T_0, \ldots, T_{r-1}\} := 1\text{-}\mathrm{Orb}(G_e)$ be the 1-orbits of the *stabilizer* G_e, ($e \in H$ unit element).
Let $\mathbb{Z}(H)$ denote the *group ring* $\langle \mathbb{Z}(H); +, *\rangle$ consisting of formal sums $\sum_{h \in H} \alpha_h h$, $\alpha_h \in \mathbb{Z}$.

Definition The submodule of $\mathbb{Z}(H)$ generated by the 1-orbits of the stabilizer G_e ($\underline{T} := \sum_{t \in T} t \in \mathbb{Z}(H)$, $T \subseteq H$)

$$\mathcal{S}(G, H) := \langle \underline{T_0}, \ldots, \underline{T_{r-1}} \rangle_{\mathbb{Z}}$$

is called the *transitivity module of* (G, H).

Theorem (*I. Schur* 1933): *$\mathcal{S}(G, H)$ is an S-ring.*

Contrary to the intuition of *I. Schur* not every S-ring is the transitivity module of a group (G, H)
– counterexamples by *H. Wielandt* (1954), e.g., for $H = \mathbb{Z}_5 \times \mathbb{Z}_5$.

Transitivity modules

Let a permutation group (G, M) (i.e., $G \leq \mathrm{Sym}(M)$) contain a regular subgroup H. Then one can assume $M = H$, i.e., $G \leq \mathrm{Sym}(H)$.
Let $\{T_0, \ldots, T_{r-1}\} := 1\text{-}\mathrm{Orb}(G_e)$ be the 1-orbits of the *stabilizer* G_e, ($e \in H$ unit element).
Let $\mathbb{Z}(H)$ denote the *group ring* $\langle \mathbb{Z}(H); +, * \rangle$ consisting of formal sums $\sum_{h \in H} \alpha_h h$, $\alpha_h \in \mathbb{Z}$.

Definition The submodule of $\mathbb{Z}(H)$ generated by the 1-orbits of the stabilizer G_e ($\underline{T} := \sum_{t \in T} t \in \mathbb{Z}(H)$, $T \subseteq H$)

$$\mathcal{S}(G, H) := \langle \underline{T_0}, \ldots, \underline{T_{r-1}} \rangle_{\mathbb{Z}}$$

is called the *transitivity module of (G, H)*.

Theorem (*I. Schur* 1933): *$\mathcal{S}(G, H)$ is an S-ring.*

Contrary to the intuition of *I. Schur* not every S-ring is the transitivity module of a group (G, H)
– counterexamples by *H. Wielandt* (1954), e.g., for $H = \mathbb{Z}_5 \times \mathbb{Z}_5$.

Transitivity modules

Let a permutation group (G, M) (i.e., $G \leq \mathrm{Sym}(M)$) contain a regular subgroup H. Then one can assume $M = H$, i.e., $G \leq \mathrm{Sym}(H)$.
Let $\{T_0, \ldots, T_{r-1}\} := 1\text{-}\mathrm{Orb}(G_e)$ be the 1-orbits of the *stabilizer* G_e, ($e \in H$ unit element).
Let $\mathbb{Z}(H)$ denote the *group ring* $\langle \mathbb{Z}(H); +, * \rangle$ consisting of formal sums $\sum_{h \in H} \alpha_h h$, $\alpha_h \in \mathbb{Z}$.

Definition The submodule of $\mathbb{Z}(H)$ generated by the 1-orbits of the stabilizer G_e ($\underline{T} := \sum_{t \in T} t \in \mathbb{Z}(H)$, $T \subseteq H$)

$$\mathcal{S}(G, H) := \langle \underline{T_0}, \ldots, \underline{T_{r-1}} \rangle_{\mathbb{Z}}$$

is called the *transitivity module of (G, H)*.

Theorem (*I. Schur* 1933): *$\mathcal{S}(G, H)$ is an S-ring.*

Contrary to the intuition of *I. Schur* not every S-ring is the transitivity module of a group (G, H)
– counterexamples by *H. Wielandt* (1954), e.g., for $H = \mathbb{Z}_5 \times \mathbb{Z}_5$.

Transitivity modules

Let a permutation group (G, M) (i.e., $G \leq \mathrm{Sym}(M)$) contain a regular subgroup H. Then one can assume $M = H$, i.e., $G \leq \mathrm{Sym}(H)$.
Let $\{T_0, \ldots, T_{r-1}\} := 1\text{-}\mathrm{Orb}(G_e)$ be the 1-orbits of the *stabilizer* G_e, ($e \in H$ unit element).
Let $\mathbb{Z}(H)$ denote the *group ring* $\langle \mathbb{Z}(H); +, * \rangle$ consisting of formal sums $\sum_{h \in H} \alpha_h h$, $\alpha_h \in \mathbb{Z}$.

Definition The submodule of $\mathbb{Z}(H)$ generated by the 1-orbits of the stabilizer G_e ($\underline{T} := \sum_{t \in T} t \in \mathbb{Z}(H),\ T \subseteq H$)

$$\mathcal{S}(G, H) := \langle \underline{T_0}, \ldots, \underline{T_{r-1}} \rangle_{\mathbb{Z}}$$

is called the *transitivity module of (G, H)*.

Theorem (*I. Schur* 1933): *$\mathcal{S}(G, H)$ is an S-ring.*

Contrary to the intuition of *I. Schur* not every S-ring is the transitivity module of a group (G, H)

– counterexamples by *H. Wielandt* (1954), e.g., for $H = \mathbb{Z}_5 \times \mathbb{Z}_5$.

Transitivity modules

Let a permutation group (G, M) (i.e., $G \leq \mathrm{Sym}(M)$) contain a regular subgroup H. Then one can assume $M = H$, i.e., $G \leq \mathrm{Sym}(H)$.
Let $\{T_0, \ldots, T_{r-1}\} := 1\text{-}\mathrm{Orb}(G_e)$ be the 1-orbits of the *stabilizer* G_e, ($e \in H$ unit element).
Let $\mathbb{Z}(H)$ denote the *group ring* $\langle \mathbb{Z}(H); +, * \rangle$ consisting of formal sums $\sum_{h\in H} \alpha_h h$, $\alpha_h \in \mathbb{Z}$.

Definition The submodule of $\mathbb{Z}(H)$ generated by the 1-orbits of the stabilizer G_e ($\underline{T} := \sum_{t\in T} t \in \mathbb{Z}(H)$, $T \subseteq H$)

$$\mathcal{S}(G, H) := \langle \underline{T_0}, \ldots, \underline{T_{r-1}} \rangle_{\mathbb{Z}}$$

is called the *transitivity module of (G, H)*.

Theorem (*I. Schur* 1933): *$\mathcal{S}(G, H)$ is an S-ring.*

Contrary to the intuition of *I. Schur* not every S-ring is the transitivity module of a group (G, H)
– counterexamples by *H. Wielandt* (1954), e.g., for $H = \mathbb{Z}_5 \times \mathbb{Z}_5$.

Transitivity modules

Let a permutation group (G, M) (i.e., $G \leq \mathrm{Sym}(M)$) contain a regular subgroup H. Then one can assume $M = H$, i.e., $G \leq \mathrm{Sym}(H)$.
Let $\{T_0, \ldots, T_{r-1}\} := 1\text{-}\mathrm{Orb}(G_e)$ be the 1-orbits of the *stabilizer* G_e, ($e \in H$ unit element).
Let $\mathbb{Z}(H)$ denote the *group ring* $\langle \mathbb{Z}(H); +, *\rangle$ consisting of formal sums $\sum_{h\in H} \alpha_h h$, $\alpha_h \in \mathbb{Z}$.

Definition The submodule of $\mathbb{Z}(H)$ generated by the 1-orbits of the stabilizer G_e ($\underline{T} := \sum_{t\in T} t \in \mathbb{Z}(H)$, $T \subseteq H$)

$$\mathcal{S}(G, H) := \langle \underline{T_0}, \ldots, \underline{T_{r-1}} \rangle_{\mathbb{Z}}$$

is called the *transitivity module of* (G, H).

Theorem (*I. Schur* 1933): *$\mathcal{S}(G, H)$ is an S-ring.*

Contrary to the intuition of *I. Schur* not every S-ring is the transitivity module of a group (G, H)
– counterexamples by *H. Wielandt* (1954), e.g., for $H = \mathbb{Z}_5 \times \mathbb{Z}_5$.

Helmut Wielandt (19.12.1910 - 14.2.2001)

"The proof given by Mr Schur is rather difficult ... Schurs's methods are nevertheless sufficient to supply a shorter proof of the more general theorem":

Theorem (H. Wielandt 1935): *If a permutation group G of degree n, where n is not a prime number, contains a regular abelian subgroup H, at least one of whose Sylow subgroups is cyclic, then G is either doubly transitive of imprimitive.*

Helmut Wielandt around 1960

From Helmut Wielandt's acceptance of membership of the Heidelberger Akademie der Wissenschaften (1961):

"But it is unmistakable that questions about finite structures are again coming to the force also in other areas of mathematics ... I am convinced that the 'finite' direction will be reunited with the mainstream in the course of the next few decades."

(q.e.d., e.g. by this conference here)

Helmut Wielandt (19.12.1910 - 14.2.2001)

"The proof given by Mr Schur is rather difficult ... Schurs's methods are nevertheless sufficient to supply a shorter proof of the more general theorem":

Theorem (*H. Wielandt* 1935): *If a permutation group G of degree n, where n is not a prime number, contains a regular abelian subgroup H, at least one of whose Sylow subgroups is cyclic, then G is either doubly transitive of imprimitive.*

Helmut Wielandt around 1960

From Helmut Wielandt's acceptance of membership of the Heidelberger Akademie der Wissenschaften (1961):

"But it is unmistakable that questions about finite structures are again coming to the force also in other areas of mathematics ... I am convinced that the 'finite' direction will be reunited with the mainstream in the course of the next few decades."

(q.e.d., e.g. by this conference here)

Helmut Wielandt (19.12.1910 - 14.2.2001)

"The proof given by Mr Schur is rather difficult ... Schurs's methods are nevertheless sufficient to supply a shorter proof of the more general theorem":

Theorem (*H. Wielandt* 1935): *If a permutation group G of degree n, where n is not a prime number, contains a regular abelian subgroup H, at least one of whose Sylow subgroups is cyclic, then G is either doubly transitive of imprimitive.*

Helmut Wielandt around 1960

From Helmut Wielandt's acceptance of membership of the Heidelberger Akademie der Wissenschaften (1961):

"But it is unmistakable that questions about finite structures are again coming to the force also in other areas of mathematics ... I am convinced that the 'finite' direction will be reunited with the mainstream in the course of the next few decades."

(q.e.d., e.g. by this conference here)

Helmut Wielandt (19.12.1910 - 14.2.2001)

"The proof given by Mr Schur is rather difficult ... Schurs's methods are nevertheless sufficient to supply a shorter proof of the more general theorem":

Theorem (*H. Wielandt* 1935): *If a permutation group G of degree n, where n is not a prime number, contains a regular abelian subgroup H, at least one of whose Sylow subgroups is cyclic, then G is either doubly transitive of imprimitive.*

Helmut Wielandt around 1960

From Helmut Wielandt's acceptance of membership of the Heidelberger Akademie der Wissenschaften (1961):

"But it is unmistakable that questions about finite structures are again coming to the force also in other areas of mathematics ... I am convinced that the 'finite' direction will be reunited with the mainstream in the course of the next few decades."

(q.e.d., e.g. by this conference here)

Helmut Wielandt (19.12.1910 - 14.2.2001)

"The proof given by Mr Schur is rather difficult ... Schurs's methods are nevertheless sufficient to supply a shorter proof of the more general theorem":

Theorem (*H. Wielandt* 1935): *If a permutation group G of degree n, where n is not a prime number, contains a regular abelian subgroup H, at least one of whose Sylow subgroups is cyclic, then G is either doubly transitive of imprimitive.*

Helmut Wielandt around 1960

From Helmut Wielandt's acceptance of membership of the Heidelberger Akademie der Wissenschaften (1961):

"But it is unmistakable that questions about finite structures are again coming to the force also in other areas of mathematics ... I am convinced that the 'finite' direction will be reunited with the mainstream in the course of the next few decades."

(q.e.d., e.g. by this conference here)

Helmut Wielandt (19.12.1910 - 14.2.2001)

"The proof given by Mr Schur is rather difficult ... Schurs's methods are nevertheless sufficient to supply a shorter proof of the more general theorem":

Theorem (*H. Wielandt* 1935): *If a permutation group G of degree n, where n is not a prime number, contains a regular abelian subgroup H, at least one of whose Sylow subgroups is cyclic, then G is either doubly transitive of imprimitive.*

Helmut Wielandt around 1960

From Helmut Wielandt's acceptance of membership of the Heidelberger Akademie der Wissenschaften (1961):
"But it is unmistakable that questions about finite structures are again coming to the force also in other areas of mathematics ... I am convinced that the 'finite' direction will be reunited with the mainstream in the course of the next few decades."

(q.e.d., e.g. by this conference here)

Outline

Some history: Burnside - Schur - Wielandt

Some more history and the characterization of (non)-Schurian S-rings

Lev Arkad'evič Kalužnin (31.1.1914 - 6.12.1990)

Миша Клин - *Лев Аркадьевич Калужнин* - Reinhard Pöschel

Kiev 1978

Schurian S-rings and Schur-groups

Lev Arkad'evič Kalužnin (influence of I. Schur and his lectures, 1933–36 in Berlin)

$\longrightarrow$ *M. Klin, R. Pöschel* (starting 1971/72)

Problems of H. Wielandt (on S-rings, e.g., how to characterize schurian S-rings among all S-rings) $\longrightarrow$

R. Pöschel, ***Untersuchungen von S-Ringen, insbesondere im Gruppenring von p-Gruppen.*** (Math. Nachr. 60(1974), 1–27)
(*Investigations of S-rings, in particular in the group ring of p-groups*)

"unfortunate" notions: *schurscher S-Ring* (*schurian S-ring*) for S-rings being the transitivity module of a group.
schurartige Gruppe (*schurian group, Schur-group*) for a group H such that each S-ring over H is schurian.

Theorem. *A finite p-group ($p \geq 5$ prime) is schurian if and only if it is cyclic.*

Corollary. *Each group with a non-cyclic p-Sylow-subgroup ($p \geq 5$ prime) is not schurian.*

Further, an *explicit description of S-rings over* $\mathbb{Z}_{p^m}$ can be given.

Schurian S-rings and Schur-groups

Lev Arkad'evič Kalužnin (influence of I. Schur and his lectures, 1933–36 in Berlin)
$\longrightarrow$ *M. Klin, R. Pöschel* (starting 1971/72)

Problems of H. Wielandt (on S-rings, e.g., how to characterize schurian S-rings among all S-rings) $\longrightarrow$

R. Pöschel, ***Untersuchungen von S-Ringen, insbesondere im Gruppenring von p-Gruppen.*** (Math. Nachr. 60(1974), 1–27)
(*Investigations of S-rings, in particular in the group ring of p-groups*)

"unfortunate" notions: *schurscher S-Ring* (*schurian S-ring*) for S-rings being the transitivity module of a group.
schurartige Gruppe (*schurian group, Schur-group*) for a group H such that each S-ring over H is schurian.

Theorem. *A finite p-group ($p \geq 5$ prime) is schurian if and only if it is cyclic.*

Corollary. *Each group with a non-cyclic p-Sylow-subgroup ($p \geq 5$ prime) is not schurian.*

Further, an *explicit description of S-rings over $\mathbb{Z}_{p^m}$* can be given.

Schurian S-rings and Schur-groups

Lev Arkad'evič Kalužnin (influence of I. Schur and his lectures, 1933–36 in Berlin)

$\longrightarrow$ *M. Klin, R. Pöschel* (starting 1971/72)

Problems of H. Wielandt (on S-rings, e.g., how to characterize schurian S-rings among all S-rings) $\longrightarrow$

R. Pöschel, ***Untersuchungen von S-Ringen, insbesondere im Gruppenring von p-Gruppen.*** (Math. Nachr. 60(1974), 1–27) (*Investigations of S-rings, in particular in the group ring of p-groups*)

"unfortunate" notions: *schurscher S-Ring* (*schurian S-ring*) for S-rings being the transitivity module of a group.
schurartige Gruppe (*schurian group, Schur-group*) for a group H such that each S-ring over H is schurian.

Theorem. *A finite p-group ($p \geq 5$ prime) is schurian if and only if it is cyclic.*

Corollary. *Each group with a non-cyclic p-Sylow-subgroup ($p \geq 5$ prime) is not schurian.*

Further, an *explicit description of S-rings over $\mathbb{Z}_{p^m}$* can be given.

Schurian S-rings and Schur-groups

Lev Arkad'evič Kalužnin (influence of I. Schur and his lectures, 1933–36 in Berlin)
$\longrightarrow$ *M. Klin, R. Pöschel* (starting 1971/72)

Problems of H. Wielandt (on S-rings, e.g., how to characterize schurian S-rings among all S-rings) $\longrightarrow$

R. Pöschel, ***Untersuchungen von S-Ringen, insbesondere im Gruppenring von p-Gruppen.*** (Math. Nachr. 60(1974), 1–27)
(*Investigations of S-rings, in particular in the group ring of p-groups*)

"unfortunate" notions: *schurscher S-Ring* (*schurian S-ring*) for S-rings being the transitivity module of a group.

schurartige Gruppe (*schurian group, Schur-group*) for a group H such that each S-ring over H is schurian.

Theorem. *A finite p-group ($p \geq 5$ prime) is schurian if and only if it is cyclic.*

Corollary. *Each group with a non-cyclic p-Sylow-subgroup ($p \geq 5$ prime) is not schurian.*

Further, an *explicit description of S-rings over $\mathbb{Z}_{p^m}$* can be given.

Schurian S-rings and Schur-groups

Lev Arkad'evič Kalužnin (influence of I. Schur and his lectures, 1933–36 in Berlin)
$\longrightarrow$ *M. Klin, R. Pöschel* (starting 1971/72)

Problems of H. Wielandt (on S-rings, e.g., how to characterize schurian S-rings among all S-rings) $\longrightarrow$

R. Pöschel, ***Untersuchungen von S-Ringen, insbesondere im Gruppenring von*** p***-Gruppen.*** (Math. Nachr. 60(1974), 1–27)
(*Investigations of S-rings, in particular in the group ring of p-groups*)

"unfortunate" notions: *schurscher S-Ring* (*schurian S-ring*) for S-rings being the transitivity module of a group.
schurartige Gruppe (*schurian group, Schur-group*) for a group H such that each S-ring over H is schurian.

Theorem. *A finite p-group ($p \geq 5$ prime) is schurian if and only if it is cyclic.*
Corollary. *Each group with a non-cyclic p-Sylow-subgroup ($p \geq 5$ prime) is not schurian.*
Further, an *explicit description of S-rings over $\mathbb{Z}_{p^m}$* can be given.

Schurian S-rings and Schur-groups

Lev Arkad'evič Kalužnin (influence of I. Schur and his lectures, 1933–36 in Berlin)
$\longrightarrow$ *M. Klin, R. Pöschel* (starting 1971/72)

Problems of H. Wielandt (on S-rings, e.g., how to characterize schurian S-rings among all S-rings) $\longrightarrow$

R. Pöschel, ***Untersuchungen von S-Ringen, insbesondere im Gruppenring von p-Gruppen.*** (Math. Nachr. 60(1974), 1–27)
(*Investigations of S-rings, in particular in the group ring of p-groups*)

"unfortunate" notions: *schurscher S-Ring* (*schurian S-ring*) for S-rings being the transitivity module of a group.
schurartige Gruppe (*schurian group, Schur-group*) for a group H such that each S-ring over H is schurian.

Theorem. *A finite p-group ($p \geq 5$ prime) is schurian if and only if it is cyclic.*

Corollary. *Each group with a non-cyclic p-Sylow-subgroup ($p \geq 5$ prime) is not schurian.*
Further, an *explicit description of S-rings over $\mathbb{Z}_{p^m}$* can be given.

Schurian S-rings and Schur-groups

Lev Arkad'evič Kalužnin (influence of I. Schur and his lectures, 1933–36 in Berlin)

$\longrightarrow$ *M. Klin, R. Pöschel* (starting 1971/72)

Problems of H. Wielandt (on S-rings, e.g., how to characterize schurian S-rings among all S-rings) $\longrightarrow$

R. Pöschel, ***Untersuchungen von S-Ringen, insbesondere im Gruppenring von p-Gruppen.*** (Math. Nachr. 60(1974), 1–27)
(*Investigations of S-rings, in particular in the group ring of p-groups*)

"unfortunate" notions: *schurscher S-Ring* (*schurian S-ring*) for S-rings being the transitivity module of a group.
schurartige Gruppe (*schurian group, Schur-group*) for a group H such that each S-ring over H is schurian.

Theorem. *A finite p-group ($p \geq 5$ prime) is schurian if and only if it is cyclic.*

Corollary. *Each group with a non-cyclic p-Sylow-subgroup ($p \geq 5$ prime) is not schurian.*

Further, an *explicit description of S-rings over $\mathbb{Z}_{p^m}$* can be given.

Schurian S-rings and Schur-groups

Lev Arkad'evič Kalužnin (influence of I. Schur and his lectures, 1933–36 in Berlin)
$\longrightarrow$ *M. Klin, R. Pöschel* (starting 1971/72)

Problems of H. Wielandt (on S-rings, e.g., how to characterize schurian S-rings among all S-rings) $\longrightarrow$

R. Pöschel, ***Untersuchungen von S-Ringen, insbesondere im Gruppenring von p-Gruppen.*** (Math. Nachr. 60(1974), 1–27)
(*Investigations of S-rings, in particular in the group ring of p-groups*)

"unfortunate" notions: *schurscher S-Ring* (*schurian S-ring*) for S-rings being the transitivity module of a group.
schurartige Gruppe (*schurian group, Schur-group*) for a group H such that each S-ring over H is schurian.

Theorem. *A finite p-group ($p \geq 5$ prime) is schurian if and only if it is cyclic.*
Corollary. *Each group with a non-cyclic p-Sylow-subgroup ($p \geq 5$ prime) is not schurian.*
Further, an *explicit description of S-rings over $\mathbb{Z}_{p^m}$* can be given.

A Galois connection

The automorphism property (concerning permutations and binary relations (graphs)) induces a Galois connection:

Answer: Theorems by *M. Krasner*, *L.A. Kalužnin* et al.:
$\operatorname{Inv}\operatorname{Aut} Q = [Q]_{\mathrm{KA}}$ *Krasner* algebra generated by Q

S Schurian $\iff$ *"closed under first order formulas"*

A Galois connection

The automorphism property (concerning permutations and binary relations (graphs)) induces a Galois connection:

$$\begin{array}{rcl}
\text{S-rings } \mathcal{S} & & \text{groups } G \geq H \\
\mathcal{S} & \longmapsto & \operatorname{Aut}\mathcal{S} \text{ automorphism group} \\
\text{transitivity module } \mathcal{S}(G,H) & \longleftarrow\!\!\dashv & (G,H)
\end{array}$$

Answer: Theorems by *M. Krasner*, *L.A. Kalužnin* et al.:
$\operatorname{Inv}\operatorname{Aut} Q = [Q]_{\mathrm{KA}}$ *Krasner* algebra generated by Q

$\mathcal{S}$ *Schurian* $\Longleftrightarrow$ *"closed under first order formulas"*

A Galois connection

The automorphism property (concerning permutations and binary relations (graphs)) induces a Galois connection:

$$\begin{array}{rcl} \text{S-rings } \mathcal{S} & & \text{groups } G \geq H \\ \mathcal{S} & \longmapsto & \operatorname{Aut}\mathcal{S} \text{ automorphism group} \\ \text{transitivity module } \mathcal{S}(G,H) & \longleftarrow & (G,H) \end{array}$$

Galois closures

$$\begin{array}{rl} \textit{Schurian} \text{ S-rings} & \textit{2-closed} \text{ permutation groups} \\ \mathcal{S}(\operatorname{Aut}\mathcal{S}, H) & G^{(2)} := \operatorname{Aut}\operatorname{Inv}^{(2)} G \\ & \quad = \operatorname{Aut} 2\text{-}\mathrm{Orb}\, G \end{array}$$

Answer: Theorems by *M. Krasner*, *L.A. Kalužnin* et al.:
$\operatorname{Inv}\operatorname{Aut} Q = [Q]_{\mathsf{KA}}$ *Krasner* algebra generated by Q
$\mathcal{S}$ *Schurian* $\iff$ *"closed under first order formulas"*

A Galois connection

The automorphism property (concerning permutations and binary relations (graphs)) induces a Galois connection:

$$\begin{array}{rcl}
\text{S-rings } \mathcal{S} & & \text{groups } G \geq H \\
\mathcal{S} & \longmapsto & \operatorname{Aut}\mathcal{S} \text{ automorphism group} \\
\text{transitivity module } \mathcal{S}(G,H) & \longleftarrow & (G,H)
\end{array}$$

Galois closures (In general: $\mathcal{S} \neq \mathcal{S}(\operatorname{Aut}\mathcal{S}, H)$, i.e., $\mathcal{S}$ is not Schurian)

$$\begin{array}{rl}
\textit{Schurian} \text{ S-rings} & \textit{2-closed} \text{ permutation groups} \\
\mathcal{S}(\operatorname{Aut}\mathcal{S}, H) & G^{(2)} := \operatorname{Aut}\operatorname{Inv}^{(2)} G \\
 & \quad = \operatorname{Aut} 2\text{-}\mathrm{Orb}\, G
\end{array}$$

Answer: Theorems by *M. Krasner*, *L.A. Kalužnin* et al.:
$\operatorname{Inv}\operatorname{Aut} Q = [Q]_{\mathrm{KA}}$ *Krasner* algebra generated by Q

$\mathcal{S}$ *Schurian* $\iff$ *"closed under first order formulas"*

A Galois connection

The automorphism property (concerning permutations and binary relations (graphs)) induces a Galois connection:

S-rings $\mathcal{S}$		groups $G \geq H$
$\mathcal{S}$	$\longmapsto$	$\operatorname{Aut}\mathcal{S}$ automorphism group
transitivity module $\mathcal{S}(G,H)$	$\longleftarrow$	(G,H)

Galois closures (In general: $\mathcal{S} \neq \mathcal{S}(\operatorname{Aut}\mathcal{S}, H)$, i.e., $\mathcal{S}$ is not Schurian)

Schurian S-rings	*2-closed* permutation groups
$\mathcal{S}(\operatorname{Aut}\mathcal{S}, H)$	$G^{(2)} := \operatorname{Aut}\operatorname{Inv}^{(2)} G$
	$= \operatorname{Aut} 2\text{-}\mathrm{Orb}\, G$

Wielandt's question: How to characterize Schurian S-rings?

Answer: Theorems by *M. Krasner*, *L.A. Kalužnin* et al.:

$\operatorname{Inv}\operatorname{Aut} Q = [Q]_{\mathrm{KA}}$ *Krasner* algebra generated by Q

$\mathcal{S}$ *Schurian* $\Longleftrightarrow$ *"closed under first order formulas"*

A Galois connection

The automorphism property (concerning permutations and binary relations (graphs)) induces a Galois connection:

S-rings $\mathcal{S}$		groups $G \geq H$
$\mathcal{S}$	$\longmapsto$	$\operatorname{Aut}\mathcal{S}$ automorphism group
transitivity module $\mathcal{S}(G,H)$	$\longleftarrow$	(G,H)

Galois closures (In general: $\mathcal{S} \neq \mathcal{S}(\operatorname{Aut}\mathcal{S}, H)$, i.e., $\mathcal{S}$ is not Schurian)

Schurian S-rings	*2-closed* permutation groups
$\mathcal{S}(\operatorname{Aut}\mathcal{S}, H)$	$G^{(2)} := \operatorname{Aut}\operatorname{Inv}^{(2)} G$
	$= \operatorname{Aut} 2\text{-}\mathrm{Orb}\, G$

Wielandt's question: How to characterize Schurian S-rings?

Answer: Theorems by *M. Krasner*, *L.A. Kalužnin* et al.:

$\operatorname{Inv}\operatorname{Aut} Q = [Q]_{\mathrm{KA}}$ *Krasner* algebra generated by Q

$\mathcal{S}$ *Schurian* $\iff$ *"closed under first order formulas"*

A Galois connection

The automorphism property (concerning permutations and binary relations (graphs)) induces a Galois connection:

$$\begin{array}{rcl} \text{S-rings } \mathcal{S} & & \text{groups } G \geq H \\ \mathcal{S} & \longmapsto & \operatorname{Aut}\mathcal{S} \text{ automorphism group} \\ \text{transitivity module } \mathcal{S}(G,H) & \longleftarrow & (G,H) \end{array}$$

Galois closures (In general: $\mathcal{S} \neq \mathcal{S}(\operatorname{Aut}\mathcal{S}, H)$, i.e., $\mathcal{S}$ is not Schurian)

$$\begin{array}{rl} \textit{Schurian} \text{ S-rings} & \textit{2-closed} \text{ permutation groups} \\ \mathcal{S}(\operatorname{Aut}\mathcal{S}, H) & G^{(2)} := \operatorname{Aut}\operatorname{Inv}^{(2)} G \\ & \quad = \operatorname{Aut} 2\text{-}\mathrm{Orb}\, G \end{array}$$

Wielandt's question: How to characterize Schurian S-rings?
Answer: Theorems by *M. Krasner*, *L.A. Kalužnin* et al.:
$\operatorname{Inv}\operatorname{Aut} Q = [Q]_{\mathsf{KA}}$ *Krasner* algebra generated by Q
$\mathcal{S}$ *Schurian* $\iff$ *"closed under first order formulas"*

A Galois connection

The automorphism property (concerning permutations and binary relations (graphs)) induces a Galois connection:

$$\begin{array}{rcl} \text{S-rings } \mathcal{S} & & \text{groups } G \geq H \\ \mathcal{S} & \longmapsto & \operatorname{Aut}\mathcal{S} \text{ automorphism group} \\ \text{transitivity module } \mathcal{S}(G,H) & \longleftarrow & (G,H) \end{array}$$

Galois closures (In general: $\mathcal{S} \neq \mathcal{S}(\operatorname{Aut}\mathcal{S}, H)$, i.e., $\mathcal{S}$ is not Schurian)

$$\begin{array}{rl} \textit{Schurian} \text{ S-rings} & \textit{2-closed} \text{ permutation groups} \\ \mathcal{S}(\operatorname{Aut}\mathcal{S}, H) & G^{(2)} := \operatorname{Aut}\operatorname{Inv}^{(2)} G \\ & \quad = \operatorname{Aut} 2\text{-}\mathrm{Orb}\, G \end{array}$$

Wielandt's question: How to characterize Schurian S-rings?
Answer: Theorems by *M. Krasner*, *L.A. Kalužnin* et al.:
$\operatorname{Inv}\operatorname{Aut} Q = [Q]_{\mathsf{KA}}$ *Krasner* algebra generated by Q

$\mathcal{S}$ *Schurian* $\iff$ *"closed under first order formulas"*

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$T_0 = \{(0,0)\}$
$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$
$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$
$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$

$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between and $2\text{-}\mathrm{Orb}\,G$:

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$T_0 = \{(0,0)\}$

$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$

$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$

$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$

$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).

But: $\mathcal{S}$ is not schurian!

Proof. Recall the 1-1-correspondence between and $2\text{-}\mathrm{Orb}\, G$:

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$T_0 = \{(0,0)\}$$
$$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$$
$$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between and 2-$\mathrm{Orb}\, G$:

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$T_0 = \{(0,0)\}$
$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$
$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$
$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$

$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between and $2\text{-}\mathrm{Orb}\,G$:

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$T_0 = \{(0,0)\}$$
$$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$$
$$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between and $2\text{-}\mathrm{Orb}\,G$:

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$T_0 = \{(0,0)\}$$
$$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$$
$$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).

But: $\mathcal{S}$ is not schurian!

Proof. Recall the 1-1-correspondence between and 2-$\mathrm{Orb}G$:

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$T_0 = \{(0,0)\}$$
$$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$$
$$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between and 2-Orb$G$:

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$T_0 = \{(0,0)\}$$
$$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$$
$$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between 1-$\mathrm{Orb}\, G_e$ and 2-$\mathrm{Orb}\, G$:

$$T = \{h \in H \mid (h,e) \in \Phi\} \longleftrightarrow \Phi = \{(h,h') \in H^2 \mid h'h^{-1} \in T\}$$

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$T_0 = \{(0,0)\}$$
$$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$$
$$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between 1-$\mathrm{Orb}\, G_0$ and 2-$\mathrm{Orb}\, G$:

$$T = \{h \in H \mid (h,0) \in \Phi\} \longleftrightarrow \Phi = \{(h,h') \in H^2 \mid h' - h \in T\}$$

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$\Phi_0 = \Delta = \{((x,y),(x,y)) \mid x,y \in \mathbb{Z}_5\}$$

$$\begin{aligned} T_0 &= \{(0,0)\} \\ T_1 &= \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0 \\ T_2 &= \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0 \\ T_3 &= \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0 \\ T_4 &= H \setminus \bigcup_{i=0}^{3} T_i \end{aligned}$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between 1-$\mathrm{Orb}\, G_0$ and 2-$\mathrm{Orb}\, G$:

$$T = \{h \in H \mid (h,0) \in \Phi\} \longleftrightarrow \Phi = \{(h,h') \in H^2 \mid h' - h \in T\}$$

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$T_0 = \{(0,0)\}$$
$$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$$
$$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

$$\Phi_0 = \Delta = \{((x,y),(x,y)) \mid x,y \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1,y),(x_2,y)) \mid x_1,x_2,y \in \mathbb{Z}_5\} \setminus \Delta$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between 1-$\mathrm{Orb}\,G_0$ and 2-$\mathrm{Orb}\,G$:

$$T = \{h \in H \mid (h,0) \in \Phi\} \longleftrightarrow \Phi = \{(h,h') \in H^2 \mid h' - h \in T\}$$

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$T_0 = \{(0,0)\}$$
$$T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0$$
$$T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0$$
$$T_4 = H \setminus \bigcup_{i=0}^{3} T_i$$

$$\Phi_0 = \Delta = \{((x,y),(x,y)) \mid x, y \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1,y),(x_2,y)) \mid x_1, x_2, y \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_2 = \{((x,y_1),(x,y_2)) \mid x, y_1, y_2 \in \mathbb{Z}_5\} \setminus \Delta$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between 1-$\mathrm{Orb}\, G_0$ and 2-$\mathrm{Orb}\, G$:

$$T = \{h \in H \mid (h,0) \in \Phi\} \longleftrightarrow \Phi = \{(h,h') \in H^2 \mid h' - h \in T\}$$

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$\begin{array}{ll}
T_0 = \{(0,0)\} & \Phi_0 = \Delta = \{((x,y),(x,y)) \mid x,y \in \mathbb{Z}_5\} \\
T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0 & \Phi_1 = \{((x_1,y),(x_2,y)) \mid x_1,x_2,y \in \mathbb{Z}_5\} \setminus \Delta \\
T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0 & \Phi_2 = \{((x,y_1),(x,y_2)) \mid x,y_1,y_2 \in \mathbb{Z}_5\} \setminus \Delta \\
T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0 & \Phi_3 = \{((x_1,y_1),(x_2,y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta \\
T_4 = H \setminus \bigcup_{i=0}^{3} T_i &
\end{array}$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between 1-$\mathrm{Orb}\,G_0$ and 2-$\mathrm{Orb}\,G$:

$$T = \{h \in H \mid (h,0) \in \Phi\} \longleftrightarrow \Phi = \{(h,h') \in H^2 \mid h' - h \in T\}$$

Example of a non-schurian S-ring

Wielandt's counterexample $H := \mathbb{Z}_5 \times \mathbb{Z}_5$

$$
\begin{array}{ll}
T_0 = \{(0,0)\} & \Phi_0 = \Delta = \{((x,y),(x,y)) \mid x, y \in \mathbb{Z}_5\} \\
T_1 = \{(x,0) \mid x \in \mathbb{Z}_5\} \setminus T_0 & \Phi_1 = \{((x_1,y),(x_2,y)) \mid x_1, x_2, y \in \mathbb{Z}_5\} \setminus \Delta \\
T_2 = \{(0,y) \mid y \in \mathbb{Z}_5\} \setminus T_0 & \Phi_2 = \{((x,y_1),(x,y_2)) \mid x, y_1, y_2 \in \mathbb{Z}_5\} \setminus \Delta \\
T_3 = \{(x,x) \mid x \in \mathbb{Z}_5\} \setminus T_0 & \Phi_3 = \{((x_1,y_1),(x_2,y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta \\
T_4 = H \setminus \bigcup_{i=0}^{3} T_i & \Phi_4 = H \times H \setminus \bigcup_{i=0}^{3} \Phi_i
\end{array}
$$

Fact: $\mathcal{S} := \langle \underline{T_0}, \underline{T_1}, \underline{T_2}, \underline{T_3}, \underline{T_4} \rangle_{\mathbb{Z}}$ is an S-ring (easy to check).
But: $\mathcal{S}$ is not schurian!
Proof. Recall the 1-1-correspondence between 1-$\mathrm{Orb}\,G_0$ and 2-$\mathrm{Orb}\,G$:

$$T = \{h \in H \mid (h,0) \in \Phi\} \longleftrightarrow \Phi = \{(h,h') \in H^2 \mid h' - h \in T\}$$

$\mathcal{S}$ is a non-schurian S-ring

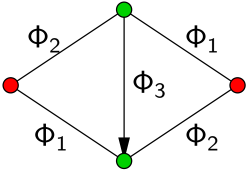
$$\begin{aligned}
\Phi_0 &= \Delta = \{(x,x) \mid x \in \mathbb{Z}_5\} \\
\Phi_1 &= \{((x_1,y),(x_2,y)) \mid x_1,x_2,y \in \mathbb{Z}_5\} \setminus \Delta \\
\Phi_2 &= \{((x,y_1),(x,y_2)) \mid x,y_1,y_2 \in \mathbb{Z}_5\} \setminus \Delta \\
\Phi_3 &= \{((x_1,y_1),(x_2,y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta \\
\Phi_4 &= H \times H \setminus \bigcup_{i=0}^{3} \Phi_i
\end{aligned}$$

$$\begin{aligned}
((x_1,y_1),(x_2,y_2)) \in \Phi :\iff \exists c_1,c_2,d_1,d_1 \\
& x_1 = c_1,\ y_1 = d_2,\ c_2 = y_2,\ d_1 = x_2, \\
& d_1 - c_1 = d_2 - c_2 \\
\iff & x_2 - x_1 = y_1 - y_2
\end{aligned}$$

$$(x,y) \in T \iff ((x,y),(0,0)) \in \Phi \iff y = -x$$

Thus, e.g., $(1,4) \in T_4 \cap T$, but $T_4 \not\subseteq T$ since $(1,2) \in T_4 \setminus T$, consequently, $\mathcal{S}$ is non-schurian. $\square$

$\mathcal{S}$ is a non-schurian S-ring



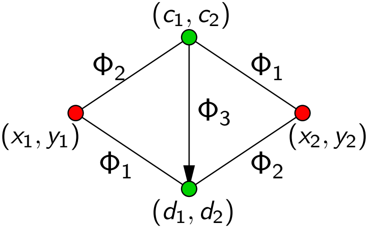
$$\Phi_0 = \Delta = \{(x,x) \mid x \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1,y),(x_2,y)) \mid x_1,x_2,y \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_2 = \{((x,y_1),(x,y_2)) \mid x,y_1,y_2 \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_3 = \{((x_1,y_1),(x_2,y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta$$
$$\Phi_4 = H \times H \setminus \bigcup_{i=0}^{3} \Phi_i$$

first order formula

$$((x_1,y_1),(x_2,y_2)) \in \Phi :\iff \exists c_1,c_2,d_1,d_1$$
$$x_1 = c_1,\ y_1 = d_2,\ c_2 = y_2,\ d_1 = x_2,$$
$$d_1 - c_1 = d_2 - c_2$$
$$\iff x_2 - x_1 = y_1 - y_2$$
$$(x,y) \in T \iff ((x,y),(0,0)) \in \Phi \iff y = -x$$

Thus, e.g., $(1,4) \in T_4 \cap T$, but $T_4 \not\subseteq T$ since $(1,2) \in T_4 \setminus T$, consequently, $\mathcal{S}$ is non-schurian. $\square$

$\mathcal{S}$ is a non-schurian S-ring



$$\Phi_0 = \Delta = \{(x, x) \mid x \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1, y), (x_2, y)) \mid x_1, x_2, y \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_2 = \{((x, y_1), (x, y_2)) \mid x, y_1, y_2 \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_3 = \{((x_1, y_1), (x_2, y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta$$
$$\Phi_4 = H \times H \setminus \bigcup_{i=0}^{3} \Phi_i$$

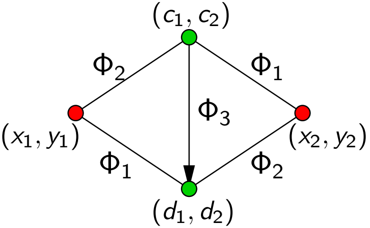
first order formula

$$\begin{aligned} ((x_1, y_1), (x_2, y_2)) \in \Phi :\Longleftrightarrow\ & \exists c_1, c_2, d_1, d_1 \\ & x_1 = c_1,\ y_1 = d_2,\ c_2 = y_2,\ d_1 = x_2, \\ & d_1 - c_1 = d_2 - c_2 \\ \Longleftrightarrow\ & x_2 - x_1 = y_1 - y_2 \end{aligned}$$

$$(x, y) \in T \iff ((x, y), (0, 0)) \in \Phi \iff y = -x$$

Thus, e.g., $(1, 4) \in T_4 \cap T$, but $T_4 \not\subseteq T$ since $(1, 2) \in T_4 \setminus T$, consequently, $\mathcal{S}$ is non-schurian. □

$\mathcal{S}$ is a non-schurian S-ring



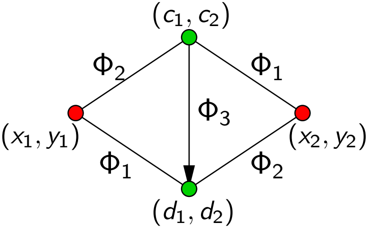
$$\Phi_0 = \Delta = \{(x, x) \mid x \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1, y), (x_2, y)) \mid x_1, x_2, y \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_2 = \{((x, y_1), (x, y_2)) \mid x, y_1, y_2 \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_3 = \{((x_1, y_1), (x_2, y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta$$
$$\Phi_4 = H \times H \setminus \bigcup_{i=0}^{3} \Phi_i$$

first order formula

$$((x_1, y_1), (x_2, y_2)) \in \Phi :\Longleftrightarrow \exists c_1, c_2, d_1, d_1$$
$$x_1 = c_1,\ y_1 = d_2,\ c_2 = y_2,\ d_1 = x_2,$$
$$d_1 - c_1 = d_2 - c_2$$
$$\Longleftrightarrow x_2 - x_1 = y_1 - y_2$$
$$(x, y) \in T \Longleftrightarrow ((x, y), (0, 0)) \in \Phi \Longleftrightarrow y = -x$$

Thus, e.g., $(1, 4) \in T_4 \cap T$, but $T_4 \not\subseteq T$ since $(1, 2) \in T_4 \setminus T$, consequently, $\mathcal{S}$ is non-schurian. □

$\mathcal{S}$ is a non-schurian S-ring



$$\Phi_0 = \Delta = \{(x, x) \mid x \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1, y), (x_2, y)) \mid x_1, x_2, y \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_2 = \{((x, y_1), (x, y_2)) \mid x, y_1, y_2 \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_3 = \{((x_1, y_1), (x_2, y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta$$
$$\Phi_4 = H \times H \setminus \bigcup_{i=0}^{3} \Phi_i$$

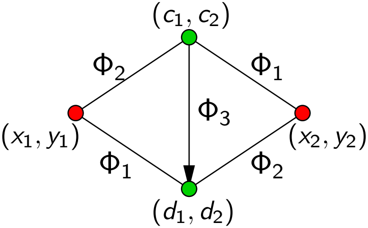
first order formula

$$\begin{aligned}((x_1, y_1), (x_2, y_2)) \in \Phi :\iff & \exists c_1, c_2, d_1, d_1 \\ & x_1 = c_1,\ y_1 = d_2,\ c_2 = y_2,\ d_1 = x_2, \\ & d_1 - c_1 = d_2 - c_2 \\ \iff & x_2 - x_1 = y_1 - y_2\end{aligned}$$

$$(x, y) \in T \iff ((x, y), (0, 0)) \in \Phi \iff y = -x$$

Thus, e.g., $(1, 4) \in T_4 \cap T$, but $T_4 \not\subseteq T$ since $(1, 2) \in T_4 \setminus T$, consequently, $\mathcal{S}$ is non-schurian. $\square$

$\mathcal{S}$ is a non-schurian S-ring



$$\Phi_0 = \Delta = \{(x, x) \mid x \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1, y), (x_2, y)) \mid x_1, x_2, y \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_2 = \{((x, y_1), (x, y_2)) \mid x, y_1, y_2 \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_3 = \{((x_1, y_1), (x_2, y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta$$
$$\Phi_4 = H \times H \setminus \bigcup_{i=0}^{3} \Phi_i$$

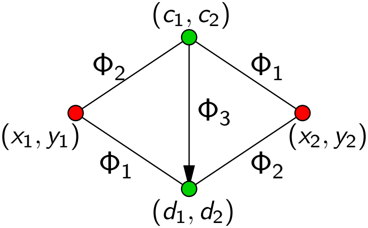
first order formula

$$\begin{aligned} ((x_1, y_1), (x_2, y_2)) \in \Phi :\iff & \exists c_1, c_2, d_1, d_1 \\ & x_1 = c_1,\ y_1 = d_2,\ c_2 = y_2,\ d_1 = x_2, \\ & d_1 - c_1 = d_2 - c_2 \\ \iff & x_2 - x_1 = y_1 - y_2 \end{aligned}$$

$$(x, y) \in T \iff ((x, y), (0, 0)) \in \Phi \iff y = -x$$

Thus, e.g., $(1, 4) \in T_4 \cap T$, but $T_4 \not\subseteq T$ since $(1, 2) \in T_4 \setminus T$, consequently, $\mathcal{S}$ is non-schurian. $\square$

$\mathcal{S}$ is a non-schurian S-ring



$$\Phi_0 = \Delta = \{(x,x) \mid x \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1,y),(x_2,y)) \mid x_1,x_2,y \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_2 = \{((x,y_1),(x,y_2)) \mid x,y_1,y_2 \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_3 = \{((x_1,y_1),(x_2,y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta$$
$$\Phi_4 = H \times H \setminus \bigcup_{i=0}^{3} \Phi_i$$

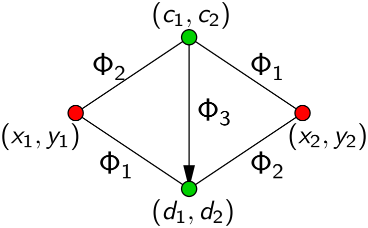
first order formula

$$\begin{aligned} ((x_1,y_1),(x_2,y_2)) \in \Phi :\iff & \exists c_1, c_2, d_1, d_1 \\ & x_1 = c_1,\ y_1 = d_2,\ c_2 = y_2,\ d_1 = x_2, \\ & d_1 - c_1 = d_2 - c_2 \\ \iff & x_2 - x_1 = y_1 - y_2 \\ (x,y) \in T \iff & ((x,y),(0,0)) \in \Phi \iff y = -x \end{aligned}$$

Thus, e.g., $(1,4) \in T_4 \cap T$, but $T_4 \not\subseteq T$ since $(1,2) \in T_4 \setminus T$,

consequently, $\mathcal{S}$ is non-schurian.

$\mathcal{S}$ is a non-schurian S-ring



$$\Phi_0 = \Delta = \{(x, x) \mid x \in \mathbb{Z}_5\}$$
$$\Phi_1 = \{((x_1, y), (x_2, y)) \mid x_1, x_2, y \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_2 = \{((x, y_1), (x, y_2)) \mid x, y_1, y_2 \in \mathbb{Z}_5\} \setminus \Delta$$
$$\Phi_3 = \{((x_1, y_1), (x_2, y_2)) \mid x_2 - x_1 = y_2 - y_1\} \setminus \Delta$$
$$\Phi_4 = H \times H \setminus \bigcup_{i=0}^{3} \Phi_i$$

first order formula

$$\begin{aligned} ((x_1, y_1), (x_2, y_2)) \in \Phi :\iff & \exists c_1, c_2, d_1, d_1 \\ & x_1 = c_1,\ y_1 = d_2,\ c_2 = y_2,\ d_1 = x_2, \\ & d_1 - c_1 = d_2 - c_2 \\ \iff & x_2 - x_1 = y_1 - y_2 \\ (x, y) \in T \iff & ((x, y), (0, 0)) \in \Phi \iff y = -x \end{aligned}$$

Thus, e.g., $(1, 4) \in T_4 \cap T$, but $T_4 \not\subseteq T$ since $(1, 2) \in T_4 \setminus T$, consequently, $\mathcal{S}$ is non-schurian. □

just a photo: We like S-rings

(Dresden 2012)


Thank you for your ATTENTION!

Happy Birthday to you Misha!