Stabilization Algorithms for Configurations

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<2018-07-02 Mo>

Outline

Numbers, Tuples

- We identify the natural number n with the set $\{0, \ldots, n-1\}$ of all smaller numbers.
- If we want to emphasize the "setness" we will write $[n]$ instead of n.
- Tuples over a set Ω are functions $x:[n] \to \Omega$
- As usual we denote the set of all function from A to B by B^A.
- In particular, the set of all n-tuples of B is $B^{[n]} = B^n$.

Kernels, equivalence relations

- Given $f : A \to B$, its kernel is the relation

$$\ker f = \{(x, y) \in A^2 \mid f(x) = f(y)\}.$$

This is an equivalence relation.

Tuples and permutations

- By $S(A)$ we denote the symmetric group of all permutations of A. Since permutations are functions they act on the left.
- If $x \in A^n$ and $\sigma \in S([n])$, then $x \circ \sigma$ is the permuted tuple: $(x \circ \sigma)(i) = x_{\sigma(i)}$.
- If x is as above, and $\varphi \in S(A)$, then $\varphi \circ x$ is the coordinatewise image of x under φ:

$$(\varphi \circ x)(i) = \varphi(x_i).$$

So,

$$\varphi \circ x = (\varphi(x_0), \ldots, \varphi(x_{n-1})).$$

Refinement of functions

- For functions $f : A \to B$, $g : A \to C$, we say $f \preceq g$ if $\ker f \subseteq \ker g$. In the case of equality we write f$\sim$ g. If $B = C$ we get a quasiorder on B^A
- If B is at most countable, then for any f:A$\to$ B there is a $g : A \to \mathbb{N}$ with f$\sim$ g. So below we can restrict ourselves to functions with codomain $\mathbb{N}$
- So we may translate between functions, equivalence relations and partitions

Binary coherent configurations

Introduction

- We recall the definition and motivation of coherent configurations.
- Later we will formalize and generalize these notions.

Colorings

- A k-coloring of Ω is a function
$$r : \Omega^k \to C$$
that assigns to each k-tuple in Ω a color from a set C.
- For $k = 2$ we think of a coloring of the edges of the complete graph on Ω.
- For now we look at the binary case and recall the notion of coherent configurations.

Binary configurations

- A binary coloring r of Ω is a *configuration* if the following properties hold:
  1. Reflexive pairs and irreflexive pairs do not share colors;
  2. If $r(x,y) = r(x',y')$, then $r(y,x) = r(y',x')$.
- Some people refer to configurations as rainbows.

Different languages

- Given a binary coloring r the preimage of each color is a binary relation on Ω.
- Hence a coloring defines a set of binary relations on Ω such that Ω^2 is its disjoint union.
- Conversely, any such system of relations defines a coloring.

Configurations as systems of relations

- In these terms we can define binary configurations as follows:
- A set S of binary relations on Ω is a configuration if
  - each relation is reflexive or irreflexive
  - if $s \in S$ then $s^* \in S$.
- Here, $s^* = \{(y, x) \mid (x, y) \in s\}$ is the inverse of s.
- We will switch freely between the languages of colorings and relations

2-homogeneous configurations

- Let G be a group acting on Ω
- The orbits of G on Ω^2 form a configuration
- We say that a configuration is *2-homogeneous* if it "comes from a group"
- More formally it means that the automorphism group acts transitively on each of the relations (better definition will follow)

Example: C_6

- Define the following configuration on $\Omega = Z_6$:
  - $R_0 = \{(x,x) | x \in \Omega\}$
  - $R_1 = \{(x,y) | x\text{-}y \in \{1,5\}\}$
  - $R_2 = \Omega^2 \setminus (R_0 \cup R_1)$
- This is a configuration.
- Does it come from a group?

Invariants

- Given a configuration we may define invariants on pairs of points.
- For example, we can count triangles of given given colors
  - Given (x,y)$\in \Omega^2$ and colors i,j, we count

$$\{z \in \Omega \mid r(x,z) = i, r(z,y) = j\}$$

- In the C_6 example this allows us to distinguish long and short diagonals

Stabilization

- Such invariants can be used to refine the given coloring
- Configurations stable under this refinement are called *coherent*
- 2-homogeneous configurations are always coherent
- The converse does not hold.

Weisfeiler-Leman

- Each coloring has a unique "smallest" coherent refinement
- We call it the *coherent closure*
- This is in turn refined by the 2-orbits of the automorphism group
- So we get a "combinatorial approximation" of the automorphism group
- The coherent closure can be computed in polynomial time, this was first described by Weisfeiler and Leman
- Several practical implementations were described by Babel, Chuvaeva, Klin, Pasechnik in the 1990's
- We might see an example of such calculations at the end of the presentation

General configurations

- We generalize the notion of coherent configurations in several aspects:
  - Instead of binary configurations we consider arbitrary arity
  - Instead of triangles we count substructures of arbitrary size.
- It is often convenient to use the language of colorings
- But what are useful generalizations of the axioms of configurations?

Plan

- We look at the defining properties of binary configurations and coherent configurations one by one
- We try to give "natural" generalizations for colorings of higher arity
- This will lead objects similar to systems of k-orbits of groups.

Reflexive/irreflexive

- The first property of binary configurations states that reflexive and irreflexive pairs have different colors
- Irreflexive pairs have a discrete kernel; reflexive pairs have a trivial kernel
- So the first condition for a k-ary coloring r is:
  - If $r(x) = r(y)$, then $\ker(x) = \ker(y)$.

Inverses

- The second property was: If two pairs have the same color, then the reverse pairs also have the same color
- For k-tuples we can apply arbitrary permutations:
  - If $r(x) = r(y)$, and $\sigma \in S_k$, then $r(x \circ \sigma) = r(y \circ \sigma)$

k-ary configurations

- Let $r : \Omega^k \to C$ be a k-coloring.
- We call r a *k-ary configuration* if the following conditions hold:
  - For $x, y \in \Omega^k$: $r(x) = r(y) \implies \ker(x) = \ker(y)$
  - For $\sigma \in S_k$, if $r(x) = r(y)$ then $r(x \circ \sigma) = r(y \circ \sigma)$.
- We call $|\Omega|$ the *order* of r; k its *arity*, and the cardinality $\left|r\left(\Omega^k\right)\right|$ of its image the *rank* of r.

Group configurations

- Let G be a group acting on Ω.
- For $x \in \Omega^k$ and $g \in G$ we have $g \circ x \in \Omega^k$.
- This defines an action of G on Ω^k.
- The orbits of this action form a k-ary configuration $(G,\Omega)^k$
- For now we call these *group configurations*

Subconfigurations

- Let r be a k-ary coloring on Ω
- Let $x \in \Omega^m$ be a tuple
- Let $x^k : [m]^k \to \Omega^k$ be the k-fold tupling of x
- Then $r \circ x^k$ is a k-ary coloring of $[m]$, the coloring r_x induced by x.

Lemma

If r is a configuration and x is one-to-one then r_x is a configuration.

Homomorphisms

- Let $W_1 = (\Omega_1, C_1, r_1)$ and $W_2(\Omega_2, C_2, r_2)$ be k-ary structures. Let $\varphi : \Omega_1 \to \Omega_2$ be a function.
    - φ is a *weak homomorphism* if for any $x, y \in \Omega_1^k$ we have $r_1(x) = r_1(y) \Longrightarrow r_2(\varphi(x)) = r_2(\varphi(y))$. We write $\varphi : W_1 \to W_2$.
    - φ is a *strong homomorphism* if $r_2 \circ \varphi = r_1$.
- A bijective strong homomorphism is an isomorphism

Homogeneity

- Let r be a k-ary configuration.
- If every isomorphism between subconfigurations of order at most m extends to an automorphism, we say that r is m-homogeneous.
- More formally: r is m-homogeneous if for any $x, y \in \Omega^m$ with $r_x = r_y$ there is an automorphism σ of r with

$$y = x \circ \sigma$$

Lemma

W is k-homogeneous iff it is a group configuration.

Extensions of vectors

- Let $n \geq m$, $x \in A^m$, $y \in A^n$. We call y an *n-extension* of x if they coincide on the first m coordinates, i.e., $x = y|_{[m]}$.
- Denote the set of all extensions of x by

$$A_x^n = \{y \in A^n \mid y|_{[m]} = x\}$$

- We denote multisets by using square brackets. E.g.,

$$[x^2 \mid x \in \mathbb{Z}, -2 \leq x \leq 2] = [0, 1, 1, 4, 4].$$

(m,t)-invariant

- Let $W = (\Omega, C, r)$ be a k-ary configuration.
- Let $t \geq m \geq k$, let $x \in \Omega^m$.
- We consider the multiset of configurations induced by all t-extensions of x.

$$W_x^t = [W_y \mid y \in \Omega_x^t]$$

Lemma

This invariant can be computed in polynomial time.

(m,t)-coherent configurations

We say that W is (m, t)-*coherent* if it is stable under this invariant.

Lemma

If $m' \leq m$ and $t' \leq t$ then any (m, t)-coherent configuration is (m', t')-coherent.

(k,t)-coherent closure

- Any k-ary configuration has a unique smallest (k,t)-coherent closure.
- This closure can be computed in time $n^{O(t)}$.
- This constitutes a Schurian polynomial approximation scheme in the sense of Evdokimov-Ponomarenko, 1999

Connection to other notions of regularity

Lemma

A k-ary configuration is coherent if and only if it is $(k, k+1)$-coherent.

- In particular, classical (binary) coherent configurations are precisely (2,3)-coherent.
- Hestenes and Higman introduced the t-vertex condition for graphs to get a stronger combinatorial characterization of rank 3 groups

Lemma

A binary configuration satisfies the t-vertex condition if and only if it is $(2, t)$-coherent.

Lemma

A k-ary configuration of order n is m-homogeneous if and only if it is (m, n)-coherent.

- So we have a family of properties for k-ary configurations which subsumes several regularity conditions considered earlier.

Implementation

- We have an implementation that computes (m,t)-coherent closures
- It still needs some optimization
- However it is a working program for this general problem
- The code will be available at
  - `http://www.github.com/sven-reichard/stabilization`

Demonstration

- Classical WL-Stabilization for Möbius ladders
- (2,4)-stabilization of Shrikhande's graph

Main question

- Are there (m,t)-coherent configurations which are not m-homogeneous, for large values of m and/or t?
- If yes, these should be rare and interesting objects.
- If not, we have solved the isomorphism problem

A related notion and examples

- Pech has introduced a similar notion for simple graphs
- His concept corresponds to (m,t)-coherence of binary configurations with three colors.
- He gives examples of (3,7)-coherent graphs arising from generalized quadrangles.

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