Coherent Configurations from Quantum Permutation Groups

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Joint work with

M. Lupini and L. Mančinska
Nonlocal Games and Quantum Permutation Groups
arXiv:1712.01820
&
A. Atserias, L. Mančinska, R. Šámal, S. Severini, and A. Varvitsiotis
Quantum and Non-signalling Graph Isomorphisms
arXiv:1611.09837

C^*-algebras

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Gelfand-Naimark Theorem

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Intuition

Think of algebras of finite dimensional complex matrices closed under conjugate transpose.

Magic unitaries

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Definition

Let $\mathcal{A}$ be a C^*-algebra. An $n \times n$ matrix $\mathcal{P} = (p_{ij})$ with $p_{ij} \in \mathcal{A}$ is a **magic unitary** if

1. $p_{ij} = p_{ij}^2 = p_{ij}^*$;
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Remark 3. The matrix $\mathcal{P}$ is unitary.

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Banica (2005): quantum automorphism groups of graphs

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Call this deformed algebra $\mathrm{C}(\Gamma^+)$, where Γ^+ is a "quantum" version of Γ

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Question. What precisely is $\mathrm{Qut}(G)$?

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- Known for vertex transitive graphs up to 11 vertices. All examples are classical or some product of classical and/or the quantum groups listed above.

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Theorem.

1. The orbitals of $\mathrm{Qut}(G)$ form a coherent configuration/algebra.
2. The matrices in this algebra are exactly those that commute with $\mathcal{P}$.

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Thus $\mathcal{C}$ is a coherent algebra.

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Theorem. Asymptotically almost surely $\mathrm{Qut}(G)$ is trivial.

Quantum Isomorphisms

Graph isomorphism as a game

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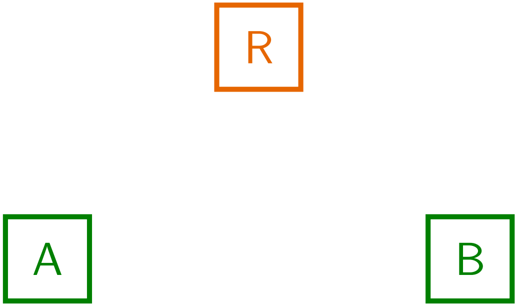

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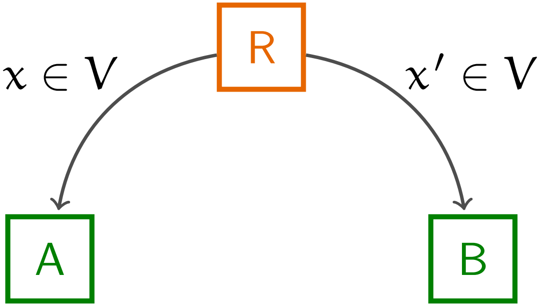

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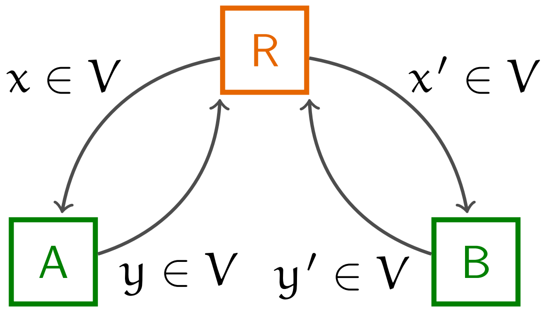

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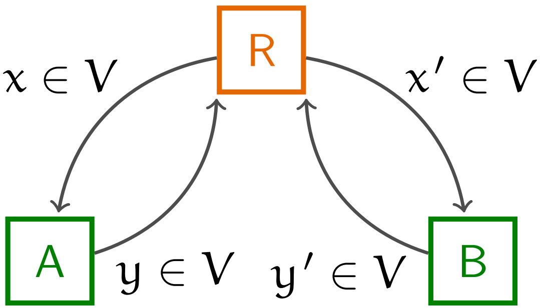

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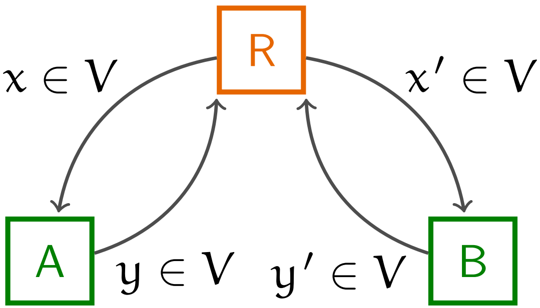

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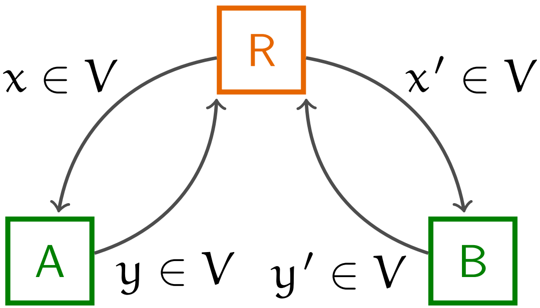

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Claim. Alice and Bob can **win** this game with probability 1 if and only if $G \cong H$.

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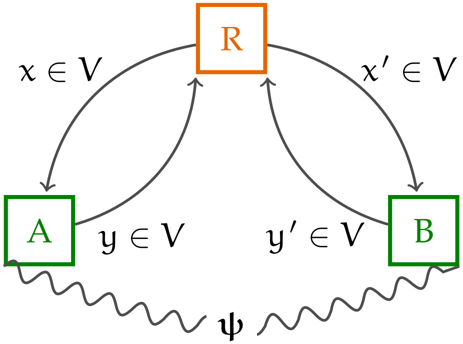

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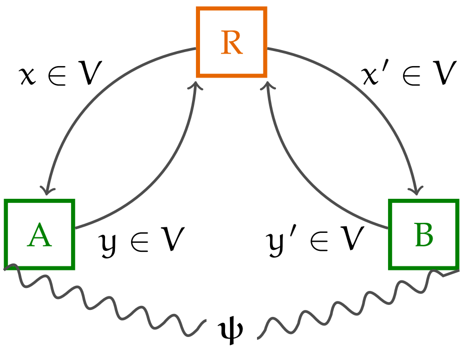

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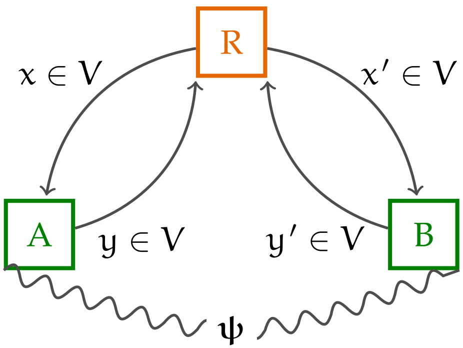

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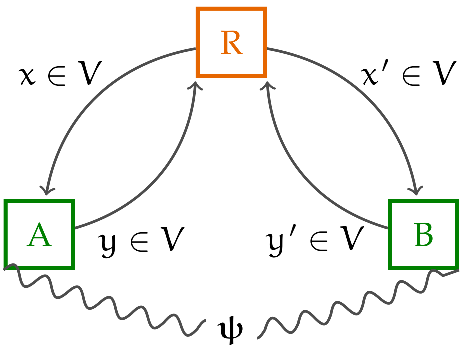

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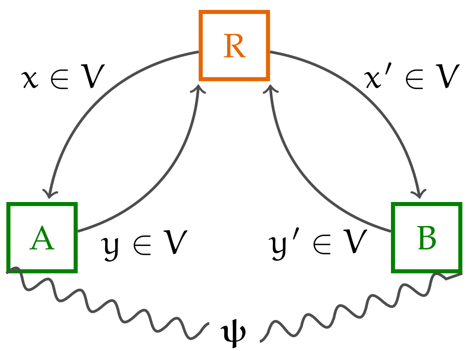

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For graphs G and H, the following are equivalent:

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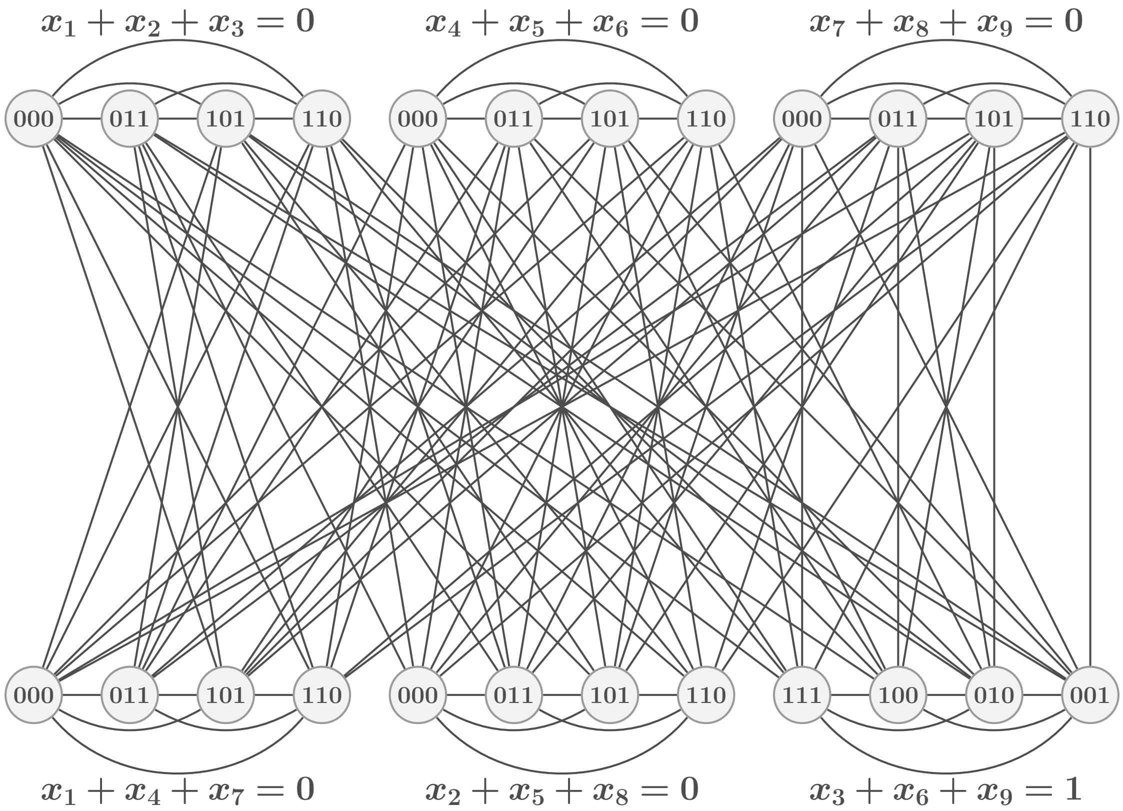
Corollary

If $G \cong_q H$, then G and H are not distinguished by the (2-dimensional) Weisfeiler-Leman algorithm.

Example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000
011
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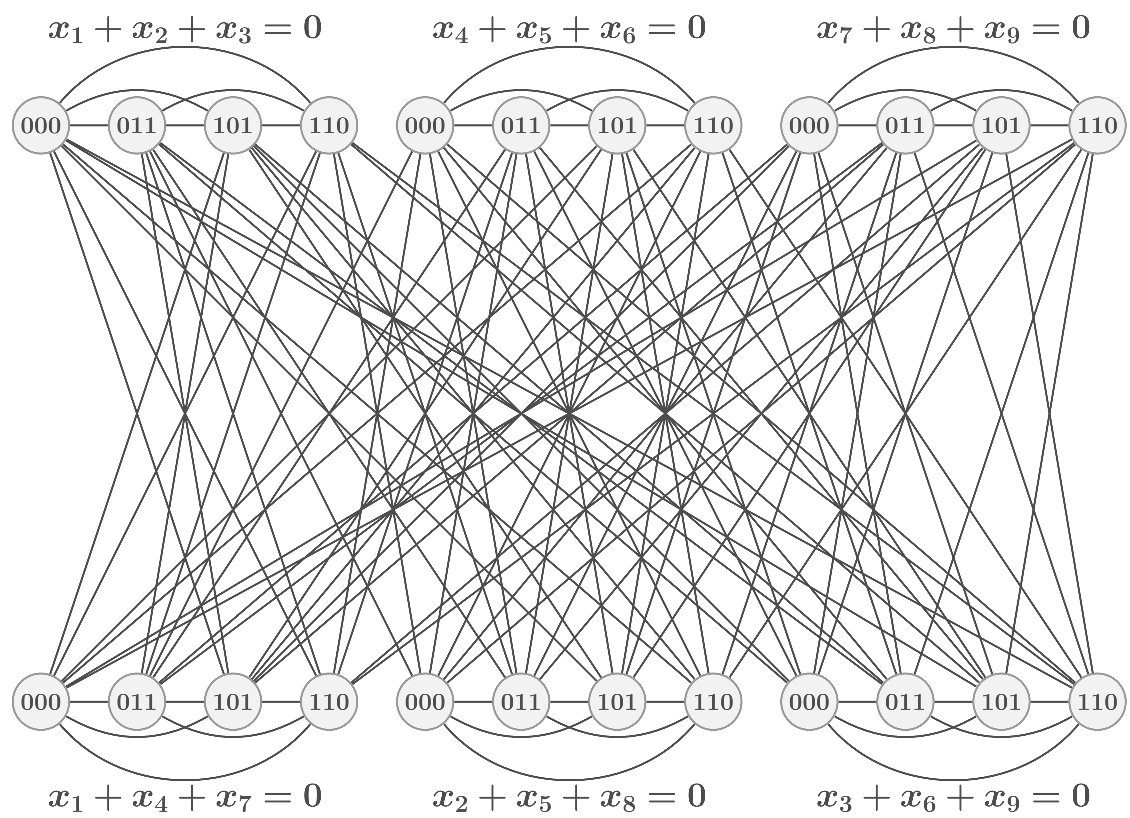

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4. Smallest pair of non-isomorphic but quantum isomorphic graphs?

Other papers on quantum isomorphisms/automorphisms

A compositional approach to quantum functions.
`arXiv:1711.07945`

The Morita theory of quantum graph isomorphisms.
`arXiv:1801.09705`

Both by Benjamin Musto, David Reutter, and Dominic Verdon of Oxford University.

Thank you!